Pseudo-Riemannian calculi and noncommutative Levi-Civita connections

Joakim Arnlind
Linköpings universitet

Worskhop Swedish Network for Algebra and Geometry
28th of September, 2018

References

Two papers together with M. Wilson:

- **Riemannian curvature of the noncommutative 3-sphere** (J. Noncommut. Geom. 2017)
- **On the Chern-Gauss-Bonnet theorem for the noncommutative 4-sphere** (J. Geom. Phys. 2017)

Introduction

- Over the last decade, a lot of progress has been made in order to understand the Riemannian aspects of noncommutative geometry.
- For the description of "quantum gravity" it is of utmost importance to understand the Riemannian structure of noncommutative space.
- We are interested in various aspects of (analogues of) the Riemannian curvature of noncommutative manifolds/algebras.
- How far one can get with a naive approach to curvature? Connection, curvature tensor, Ricci and scalar curvature?
- Our approach is naive in the sense that we collect what we need in order to prove what we want. This sheds light on the question how much one can expect in the general situation.
- In particular: What kind of algebraic structure guarantees the existence of a (torsionfree, metric) Levi-Civita connection?

General idea

In Riemannian geometry, the main (algebraic) objects are:

Algebra of functions $\mathcal{A}$, vector fields M, the metric g.

- M – (finitely generated) projective $\mathcal{A}$-module. (One can multiply a vector field with a function to get a new vector field). Serre-Swan theorem.
- g – The metric is a map $g : M \times M \to \mathcal{A}$ (giving the inner product between tangent vectors at each point).

Finally, there is another algebraic structure which is important: $\mathrm{Der}(\mathcal{A})$ – the set of (algebraic) derivations of the algebra $\mathcal{A}$.

Vector fields and derivations

Recall that a derivation $d : \mathcal{A} \to \mathcal{A}$ is a linear map such that

$$d(ab) = a(db) + (da)b$$

for all $a, b \in \mathcal{A}$. In differential geometry, every vector field acts as a derivation of the algebra of functions. In local coordinates:

$$X = X^i \frac{\partial}{\partial x^i} \quad \Longrightarrow \quad X(f) = X^i \frac{\partial f}{\partial x^i}$$

A well-known theorem in differential geometry states that there is a one-to-one correspondence between the vector fields and the derivations of the algebra of functions. That is, there is an isomorphism of modules

$$\varphi : \mathrm{Der}(\mathcal{A}) \to M.$$

Connections

Recall that a connection (on the tangent bundle) is a map $\nabla : M \times M \to M$ such that

1. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
2. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$
3. $\nabla_{fX} Y = f\nabla_X Y$
4. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$ $\boxed{X(f)Y}$

for $f \in \mathcal{A}$ and $X, Y, Z \in M$.

In the fourth requirement, we need a vector field X to act as a derivation. This is a notion that we would to generalize to the noncommutative setting.

(Note that there are of course formulations of differential forms and connections in noncommutative geometry that avoid derivations, but in many cases one would like a "derivation based calculus".)

Noncommutative algebraic version?

The idea is to naively copy the preceeding algebraic structures of Riemannian geometry to the case when the algebra $\mathcal{A}$ is noncommutative. Thus, we want to choose

- $\mathcal{A}$ – noncommutative $*$-algebra (complex valued functions)
- M – projective (right) $\mathcal{A}$-module (vector fields)
- h – $\mathcal{A}$-bilinear map $h : M \times M \to \mathcal{A}$ (metric)
- $\nabla : \mathrm{Der}(\mathcal{A}) \times M \to M$ (connection)

In this context there is an important difference between commutative and noncommutative algebras: $\mathrm{Der}(\mathcal{A})$ is in general not a module if $\mathcal{A}$ is a noncommutative algebra. That is, if one multiplies a derivation with an algebra element, it is no longer a derivation.

Thus, it doesn't make sense to assume that M and $\mathrm{Der}(\mathcal{A})$ are isomorphic as $\mathcal{A}$-modules.

Metric modules

Let us now be more precise, and introduce the concepts leading to a definition of "Pseudo-Riemannian calculus".

A hermitian form on the right $\mathcal{A}$-module M is a map $h : M \times M \to \mathcal{A}$ such that for $a \in \mathcal{A}$ and $U, V, W \in M$

$$
\begin{aligned}
h(U, V + W) &= h(U, V) + h(U, W)\\
h(U, Va) &= h(U, V)a\\
h(U, V)^* &= h(V, U).
\end{aligned}
$$

We say that h is nondegenerate if $h(U, V) = 0$ for all $V \in M$ implies that $U = 0$.

If h is a nondegenerate hermitian form on M, we say that the pair (M, h) is a (right) metric $\mathcal{A}$-module.

Real metric calculus

Definition

Let (M, h) be a (right) metric $\mathcal{A}$-module, let $\mathfrak{g} \subseteq \mathrm{Der}(\mathcal{A})$ be a (real) Lie algebra of hermitian derivations and let $\varphi : \mathfrak{g} \to M$ be a $\mathbb{R}$-linear map. Denoting the pair $(\mathfrak{g}, \varphi)$ by $\mathfrak{g}_\varphi$, the triple $(M, h, \mathfrak{g}_\varphi)$ is called a *real metric calculus* if it holds that

1. the image $M_\varphi = \varphi(\mathfrak{g})$ generates M as a (right) $\mathcal{A}$-module,
2. $h(E_1, E_2)^* = h(E_1, E_2)$ for all $E_1, E_2 \in M_\varphi$.

- We do not need to consider all derivations; we assume $\mathfrak{g}$ is a sub-(Lie)algebra of $\mathrm{Der}(\mathcal{A})$.
- Every derivation corresponds to a vector field via $\varphi : \mathfrak{g} \to M$. However, there are vector fields that do not correspond to a derivation.
- The vector fields that correspond to derivations generate all vector fields.

Real connection calculus

Let us now add a connection to the previous data.

Definition

Let $(M, h, \mathfrak{g}_\varphi)$ be a real metric calculus and let $\nabla : \mathfrak{g} \times M \to M$ denote an affine connection on M. If it holds that

$$h(\nabla_d E_1, E_2) = h(\nabla_d E_1, E_2)^*$$

for all $E_1, E_2 \in M_\varphi$ and $d \in \mathfrak{g}$ then $(M, h, \mathfrak{g}_\varphi, \nabla)$ is called a *real connection calculus*.

Pseudo-Riemannian calculus

The Levi-Civita connection is metric and torsionfree, so let us introduce these concepts in our framework.

Definition

Let $(M, h, \mathfrak{g}_\varphi, \nabla)$ be a real connection calculus over M. The calculus is *metric* if

$$d\big(h(U, V)\big) = h\big(\nabla_d U, V\big) + h\big(U, \nabla_d V\big)$$

for all $d \in \mathfrak{g}$, $U, V \in M$, and *torsionfree* if

$$\nabla_{d_1}\varphi(d_2) - \nabla_{d_2}\varphi(d_1) - \varphi\big([d_1, d_2]\big) = 0$$

for all $d_1, d_2 \in \mathfrak{g}$. A metric and torsionfree real connection calculus over M is called a *pseudo-Riemannian calculus over M*.

Uniqueness of the pseudo-Riemannian calculus

Given a real metric calculus, there is no guarantee that one may find a torsionfree and metric connection. So far the assumptions on the algebraic objects involved are close to none, and it is not surprising that existence can not be guaranteed.

However, if such a connection exists, it is unique:

Proposition

Let $(M, h, \mathfrak{g}_\varphi)$ be a real metric calculus over M. Then there exists at most one connection ∇ on M, such that $(M, h, \mathfrak{g}_\varphi, \nabla)$ is a pseudo-Riemannian calculus (i.e., such that ∇ is a real, torsionfree and metric connection).

The noncommutative torus

The noncommutative torus T_q^2 is defined via two unitary generators Z, W satisfying $WZ = qZW$, where $|q| = 1$. Introduce

$$X^1 = \frac{1}{2\sqrt{2}}(U^* + U) \qquad X^2 = \frac{i}{2\sqrt{2}}(U^* - U)$$
$$X^3 = \frac{1}{2\sqrt{2}}(V^* + V) \qquad X^4 = \frac{i}{2\sqrt{2}}(V^* - V)$$

Let $\mathfrak{g}$ be the Lie algebra generated by the two canonical derivations δ_1, δ_2 on T_q^2. M is the submodule of $(T_q^2)^4$ generated by

$$E_1 = (-X^2, X^1, 0, 0) \qquad E_2 = (0, 0, -X^4, X^3)$$

Define $\varphi : \mathfrak{g} \to M$ by $\varphi(\delta_i) = E_i$ for $i = 1, 2$. One may choose a metric as $h(E_i, E_j) = h\delta_{ij}$, where $h \in T_q^2$ is hermitian.

The noncommutative 3-sphere

We consider the 3-sphere as defined by K. Matsumoto: Let S^3_θ be the $*$-algebra generated by two **normal** elements Z, W satisfying

$$WZ = qZW \qquad W^*Z = \bar{q}ZW^* \qquad WW^* + ZZ^* = \mathbb{1},$$

and introduce

$$X^1 = \frac{1}{2}(Z + Z^*) \qquad X^2 = \frac{1}{2i}(Z - Z^*)$$
$$X^3 = \frac{1}{2}(W + W^*) \qquad X^4 = \frac{1}{2i}(W - W^*),$$

implying $(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \mathbb{1}$. Normality of Z, W is equivalent to $[X^1, X^2] = [X^3, X^4] = 0$.

The module of vector fields

Let M be the submodule of $(S^3_\theta)^4$ generated by

$$
\begin{aligned}
E_1 &= (-X^2, X^1, 0, 0) \\
E_2 &= (0, 0, -X^4, X^3) \\
E_3 &= (X^1|W|^2, X^2|W|^2, -X^3|Z|^2, -X^4|Z|^2),
\end{aligned}
$$

where $|Z|^2 = ZZ^*$ and $|W|^2 = WW^*$.

Let $\mathfrak{g}$ be the Lie algebra generated by the derivations

$$
\begin{aligned}
\partial_1(Z) &= iZ & \partial_1(W) &= 0 \\
\partial_2(Z) &= 0 & \partial_2(W) &= iW \\
\partial_3(Z) &= Z|W|^2 & \partial_3(W) &= -W|Z|^2,
\end{aligned}
$$

and $[\partial_a, \partial_b] = 0$ for $a, b = 1, 2, 3$. Furthermore, set $\varphi(\partial_a) = E_a$.

Define

$$h(U,V) = (U^a)^* h_{ab} V^b$$

where

$$h_{ab} = \sum_{k=1}^{4} (E_a^k)^* E_b^k = \begin{pmatrix} |Z|^2 & 0 & 0 \\ 0 & |W|^2 & 0 \\ 0 & 0 & |Z|^2|W|^2 \end{pmatrix}.$$

The above data defines a real metric calculus, and one may compute the (unique) Levi-Civita connection as

$$\begin{aligned} &\nabla_{\partial_1} E_1 = -E_3 \quad && \nabla_{\partial_2} E_2 = E_3 \quad && \nabla_{\partial_3} E_3 = E_3(|W|^2 - |Z|^2) \\ &\nabla_{\partial_1} E_2 = 0 && \nabla_{\partial_1} E_3 = E_1|W|^2 && \nabla_{\partial_2} E_3 = -E_2|Z|^2. \end{aligned}$$

Curvature

One may proceed to compute the curvature operators

$$R(\partial_a,\partial_b)U = \nabla_{\partial_a}\nabla_{\partial_b}U - \nabla_{\partial_b}\nabla_{\partial_a}U - \nabla_{[\partial_a,\partial_b]}U$$

$$R(\partial_1,\partial_2) = \begin{pmatrix} 0 & |W|^2 & 0 \\ -|Z|^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(\partial_1,\partial_3) = \begin{pmatrix} 0 & 0 & |Z|^2|W|^2 \\ 0 & 0 & 0 \\ -|Z|^2 & 0 & 0 \end{pmatrix}$$

$$R(\partial_2,\partial_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & |Z|^2|W|^2 \\ 0 & -|W|^2 & 0 \end{pmatrix}$$

The noncommutative 4-sphere

In the same way, one may consider the noncommutative 4-sphere: For $\theta \in [0,1)$, we let S^4_θ denote the unital $*$-algebra (over $\mathbb{C}$) generated by Z, W and T, satisfying the relations

$$
\begin{aligned}
&WZ = qZW \qquad W^*Z = \bar{q}ZW^* \\
&ZZ^* + WW^* + T^2 = \mathbb{1} \\
&T^* = T \qquad [T,Z] = [T,W] = [W,W^*] = [Z,Z^*] = 0,
\end{aligned}
\tag{1}
$$

where $q = e^{i2\pi\theta}$. One can construct a pseudo-Riemannian calculus over the noncommutative 4-sphere with respect to the metrics:

$$
(h_{ab}) = \delta \begin{pmatrix} |Z|^2(\mathbb{1} - T^2)^2 & 0 & 0 & 0 \\ 0 & |W|^2(\mathbb{1} - T^2)^2 & 0 & 0 \\ 0 & 0 & |Z|^2|W|^2(\mathbb{1} - T^2) & 0 \\ 0 & 0 & 0 & \mathbb{1} - T^2 \end{pmatrix}
$$

where $\delta \in S^4_\theta$ is assumed to be hermitian, central and regular.

The classical Gauss-Bonnet theorem

The classical Gauss-Bonnet theorem states that for a compact surface Σ without boundary

$$\int_\Sigma K\sqrt{g} = 2\pi\chi(\Sigma)$$

where K denotes the Gaussian curvature (half the scalar curvature) and χ denotes the Euler characteristic of Σ.

The importance of such theorem is that it connects metric information (the curvature) with topological information (the Euler characteristic). Hence, the value of the above integral is the same for *all* possible Riemannian metrics on Σ.

Over the last decade, people have worked hard on understanding similar statements for the noncommutative torus.

The Chern-Gauss Bonnet theorem

For even dimensional manifolds (the odd dimensional case is trivial), it is no longer the scalar curvature which gives a topological integral, but rather the Pfaffian of the curvature form. For the pseudo-Riemannian calculus constructed on the noncommutative 4-sphere, one can make sense of this and prove the following.

Theorem

Let $\delta(T)$ be an invertible polynomial in the localization of the center of S^4_θ. Then (modulo technical assumptions)

$$\chi(S^4_\theta) = \frac{1}{32\pi^2}\tau_{\delta,\mathrm{loc}}\left(R^{abcd}R_{abcd} - 4\,\mathrm{Ric}_{ab}\,\mathrm{Ric}^{ab} + S^2\right) = 2.$$

Where $\tau_{\delta,\mathrm{loc}}$ denotes an integral/trace on S^4_θ.

Summary

- We have constructed a calculus over an algebra, which involves choosing an appropriate module with an hermitian form (M, h), and a Lie algebra of derivations $\mathfrak{g}$ together with a map $\varphi : \mathfrak{g} \to M$, associating a "vector field" to each derivation.
- In this calculus one may discuss torsionfree and metric connections on M, and prove that such a connection is unique if it exists.
- We hope that these naive investigations can shed light on what kind of properties one can expect when considering curvature of noncommutative manifolds.
- We have proven a simple analogue of the Chern-Gauss-Bonnet theorem for the noncommutative 4-sphere. Of course, it depends on a number of choices and particular properties of the 4-sphere, but we hope it can still teach us something about the general case.

Thanks for listening!