Skew inverse semigroup rings

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BTH

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Outline

1. Inverse semigroups
2. Skew inverse semigroup rings
3. Steinberg algebras

Inverse semigroups

Definition

A semigroup S is said to be an *inverse semigroup*, if for each $s \in S$ there is a unique element $s^* \in S$ such that

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*$$

hold.

Examples

- Any group (with $g^* = g^{-1}$)
- Any semilattice (semigroup of commuting idempotents, with $s^* = s$)
- $S = \mathbb{Q}$ with multiplication

The symmetric inverse semigroup

Example (The symmetric inverse semigroup on X)

- Let X be a non-empty set.
- Put
$$\mathcal{I}_X := \{ \text{ bijections } A \to B \mid A, B \text{ subsets of } X \}$$
- Equip $\mathcal{I}_X$ with the binary operation "function composition whenever possible".
- $\mathcal{I}_X$ is an inverse semigroup!

Theorem (Wagner-Preston)

For each inverse semigroup S, there is some set X such that S can be realized as an inverse subsemigroup of $\mathcal{I}_X$.

An involution on S

Take any $s \in S$ and fix it. We have

$$\begin{cases} ss^*s &= s \\ s^*ss^* &= s^*. \end{cases} \tag{1}$$

Since s is fixed, so is s^* and we may consider the set of equations

$$\begin{cases} s^*xs^* &= s^* \\ xs^*x &= x. \end{cases} \tag{2}$$

Notice that $x = s$ is a solution. Thus, $s = (s^*)^*$.

Remark

One can also show that $(st)^* = t^*s^*$, for any $s, t \in S$.

$E(S)$, the set of idempotents in S

Recall: $E(S) := \{u \in S \mid u^2 = u\}$

Example

For any $s \in S$, the element ss^* is an idempotent. Indeed,

$$(ss^*)(ss^*) = (ss^*s)s^* = ss^*.$$

Remark

Take any idempotent $u = u^2 \in E(S)$ and fix it.

- Consider this set of equations: $\begin{cases} uxu & = & u \\ xux & = & x \end{cases}$
- Clearly, $x = u$ is a solution. Thus, the unique solution is $u^* = x = u$.

$E(S)$ is a commutative subsemigroup of S

Let $u, v \in E(S)$ be idempotents, i.e. $u = u^2$ and $v = v^2$.

$$(uv)(v(uv)^*u)(uv) = uv^2(uv)^*u^2v = uv(uv)^*uv = uv$$

and

$$\begin{aligned}(v(uv)^*u)(uv)(v(uv)^*u) &= v(uv)^*u^2v^2(uv)^*u = v((uv)^*uv(uv)^*)u \\ &= v(uv)^*u\end{aligned}$$

Hence, $(uv)^* = v(uv)^*u$.

$$(uv)^*(uv)^* = v(uv)^*uv(uv)^*u = v((uv)^*uv(uv)^*)u = v(uv)^*u = (uv)^*$$

Thus, $(uv)^* = uv$. By symmetry, vu is also idempotent and we get

$$(uv)(vu)(uv) = uv^2u^2v = uvuv = uv \text{ and } (vu)(uv)(vu) = vuvu = vu$$

Thus, $vu = (uv)^* = uv$.

A partial order on $E(S)$ and S

Definition ($\leq$ on $E(S)$)

For $u, v \in E(S)$ we put

$$u \leq v \text{ if } u = uv.$$

Definition ($\leq$ on S)

For $s, t \in S$ we put

$$s \leq t \text{ if } s = et$$

for some idempotent e.

Remark

- $s \leq t \iff s = ss^*t \iff s = ts^*s$
- If S is unital with identity element 1, then $u \leq 1$ for any $u \in E(S)$, since $u = u \cdot 1$.
- If $s \leq t$ and $a \leq b$, then $sa \leq tb$.

Partial actions of inverse semigroups

Definition

A *partial action of an inverse semigroup* S *on a ring* $\mathcal{A}$ is a collection of ideals $\{D_s\}_{s \in S}$ of $\mathcal{A}$ and ring isomorphisms $\{\pi_s : D_{s^*} \to D_s\}_{s \in S}$ such that, for any $s, t \in S$, the following three assertions hold:

1. $\mathcal{A}$, viewed as an additive group, is generated by the set $\bigcup_{e \in E(S)} D_e$;
2. $\pi_s(D_{s^*} \cap D_t) = D_s \cap D_{st}$;
3. $\pi_s(\pi_t(x)) = \pi_{st}(x)$, for all $x \in D_{t^*} \cap D_{t^* s^*}$.

Some properties

Proposition

Let $(\{\pi_s\}_{s\in S}, \{D_s\}_{s\in S})$ *be a partial action of an inverse semigroup* S *on a ring* $\mathcal{A}$. *The following assertions hold:*

a. $D_s \subseteq D_{ss^*}$, *for any* $s \in S$;
b. $\pi_e = \mathrm{id}_{D_e}$, *for any* $e \in E(S)$;
c. $\pi_s^{-1} = \pi_{s^*}$, *for any* $s \in S$;
d. *Let* $s, t \in S$. *If* $s \leq t$, *then* $D_s \subseteq D_t$;
e. *Let* $s, t \in S$. *If* $s \leq t$, *then* $\pi_s(x) = \pi_t(x)$, *for all* $x \in D_{s^*}$.
f. *If* S *is unital, then* $D_1 = \mathcal{A}$ *and* $\pi_1 = \mathrm{id}_{\mathcal{A}}$;

Skew inverse semigroup rings: Step 1

Given a partial action $(\{\pi_s\}_{s\in S}, \{D_s\}_{s\in S})$ of S on a ring $\mathcal{A}$, we construct the corresponding *skew inverse semigroup ring* $\mathcal{A} \rtimes_\pi S$ in three steps.

1. First we consider the set

$$\mathcal{L} = \left\{ \sum_{s\in S}^{\text{finite}} a_s\delta_s \;\middle|\; a_s \in D_s \right\}$$

where δ_s, for $s \in S$, is a formal symbol. We equip $\mathcal{L}$ with component-wise addition and with a multiplication defined as the linear extension of the rule

$$(a_s\delta_s)(b_t\delta_t) = \pi_s(\pi_{s^*}(a_s)b_t)\delta_{st}.$$

Remark

- Under mild assumptions on $\mathcal{A}$, $\mathcal{L}$ will be an associative ring.
- $\mathcal{L}$ is a generalization of a *partial skew group ring*.

Skew inverse semigroup rings: Step 2

2. Then, we consider the ideal

$$\mathcal{N} = \langle a\delta_r - a\delta_s \mid r,s \in S,\ r \leq s \text{ and } a \in D_r \rangle,$$

i.e. $\mathcal{N}$ is the ideal of $\mathcal{L}$ generated by all elements of the form $a\delta_r - a\delta_s$, where $r \leq s$ and $a \in D_r$.

Lemma

The ideal $\mathcal{N}$ is equal to the additive group generated by the set $\{a\delta_r - a\delta_s \mid r,s \in S,\ r \leq s \text{ and } a \in D_r\}$.

Skew inverse semigroup rings: Step 3

3. Finally, we define the corresponding *skew inverse semigroup ring* as the quotient ring
$$\mathcal{A} \rtimes_\pi S := \mathcal{L}/\mathcal{N}.$$
Elements of $\mathcal{A} \rtimes_\pi S$ will be written as $\overline{x}$, where $x \in \mathcal{L}$.

Remark

Any partial skew group ring $\mathcal{A} \star_\alpha G$ can be realized as a skew inverse semigroup ring over the *Exel semigroup* of G.

Bad news

$\mathcal{A} \rtimes_\pi S$ does not have any natural (or useful) gradation!

The diagonal of $\mathcal{A} \rtimes_\pi S$

We define the *diagonal* of $\mathcal{A} \rtimes_\pi S$ as the following set:

$$\mathcal{D} = \left\{ \sum_{i=1}^{n} \overline{a_i \delta_{e_i}} \;\middle|\; n \in \mathbb{Z}_+,\ e_i \in E(S),\ a_i \in D_{e_i} \right\}$$

Proposition

Let $\mathcal{A}$ be a commutative and associative ring. Then $\mathcal{A}$ is embedded in $\mathcal{A} \rtimes_\pi S$ and is isomorphic to $\mathcal{D}$, which is a commutative subring of $\mathcal{A} \rtimes_\pi S$.

Ideals of $\mathcal{A} \rtimes_\pi S$

Theorem (Beuter + Gonçalves + Ö + Royer)

Let $\mathcal{A}$ be an associative and commutative ring. Then $\mathcal{A}$ is a maximal commutative subring of $\mathcal{A} \rtimes_\pi S$ if, and only if, $\mathcal{J} \cap \mathcal{A} \neq \{0\}$ for each non-zero ideal $\mathcal{J}$ of $\mathcal{A} \rtimes_\pi S$.

Simplicity of $\mathcal{A} \rtimes_\pi S$

Definition

An ideal I of $\mathcal{A}$ is S-*invariant* if $\pi_s(I \cap D_{s^*}) \subseteq I$ holds for each $s \in S$. The ring $\mathcal{A}$ is said to be S-*simple* if $\mathcal{A}$ has no non-zero S-invariant proper ideal.

Theorem (Beuter + Gonçalves + Ö + Royer)

If $\mathcal{A}$ is an associative and commutative ring, then the following two assertions are equivalent:

1. *The skew inverse semigroup ring $\mathcal{A} \rtimes_\pi S$ is simple;*
2. *$\mathcal{A}$ is S-simple, and $\mathcal{A} \cong \mathcal{D}$ is a maximal commutative subring of $\mathcal{A} \rtimes_\pi S$.*

Groupoids

Definition

A *groupoid* is a set $\mathcal{G}$ endowed with a product map

$$\mathcal{G}^{(2)} \ni (a,b) \mapsto ab \in \mathcal{G},$$

(where $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$ is the set of *composable pairs*) and an involution map

$$\mathcal{G} \ni a \mapsto a^{-1} \in \mathcal{G}$$

for which the following two assertions hold:

- if (a,b) and (b,c) are in $\mathcal{G}^{(2)}$, then so are (ab,c) and (a,bc) and the equation $(ab)c = a(bc)$ holds;
- for each $a \in \mathcal{G}$ we have $(a,a^{-1}) \in \mathcal{G}^{(2)}$ and, if $(b,c) \in \mathcal{G}^{(2)}$ then $b^{-1}(bc) = c$ and $(bc)c^{-1} = b$.

Groupoids, cont'd

Definition

Let $\mathcal{G}$ be a groupoid and let $a \in \mathcal{G}$ be arbitrary.

- The *range* map:
  $r(a) := aa^{-1}$
- The *source* map:
  $s(a) := a^{-1}a$
- The *unit space* of $\mathcal{G}$ is
  $\mathcal{G}^{(0)} := \{b \in \mathcal{G} \mid b \in r(\mathcal{G}) = s(\mathcal{G})\}$.
- For $B, C \subseteq \mathcal{G}$, we define
  - $BC := \{bc \in \mathcal{G} \mid b \in B, c \in C$ and $s(b) = r(c)\}$
  - $B^{-1} := \{b^{-1} \mid b \in B\}$

Remark

$(b, c) \in \mathcal{G}^{(2)}$ if, and only if, $s(b) = r(c)$.

Groupoids, cont'd

Definition

- A *bisection* in $\mathcal{G}$ is a subset $B \subseteq \mathcal{G}$ such that the restrictions of r and s to B are both injective.
- A topological groupoid $\mathcal{G}$ is said to be *étale* if $\mathcal{G}^{(0)}$ is locally compact and Hausdorff, and its source map is a local homeomorphism from $\mathcal{G}$ to $\mathcal{G}^{(0)}$.
- An étale groupoid $\mathcal{G}$ is said to be *ample* if $\mathcal{G}$ has a basis of compact bisections.

We shall only be interested in Hausdorff and ample groupoids!

Remark

One can show that a Hausdorff étale groupoid is ample if, and only if, $\mathcal{G}^{(0)}$ is totally disconnected.

Groupoids, cont'd

Definition

- A subset U of the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is *invariant* if $s(b) \in U$ implies $r(b) \in U$.
- We call $\mathcal{G}$ *minimal* if $\mathcal{G}^{(0)}$ has no nontrivial open invariant subset.
- The *isotropy subgroupoid of* $\mathcal{G}$ is the set

$$\mathrm{Iso}(\mathcal{G}) := \{b \in \mathcal{G} \mid r(b) = s(b)\}.$$

- A Hausdorff and ample groupoid $\mathcal{G}$ is said to be *effective* if the interior of $\mathrm{Iso}(\mathcal{G})$ is $\mathcal{G}^{(0)}$, or equivalently, for every non-empty compact bisection $B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$, there exists some $b \in B$ such that $s(b) \neq r(b)$.

The Steinberg algebra $A_R(\mathcal{G})$

Definition

Let R be a unital commutative ring and let $\mathcal{G}$ be a Hausdorff and ample groupoid. The *Steinberg algebra* $A_R(\mathcal{G})$ is the collection of compactly supported locally constant functions from $\mathcal{G}$ to R with pointwise addition, and convolution product

$$(f * g)(b) = \sum_{r(c)=r(b)} f(c)g(c^{-1}b) = \sum_{cd=b} f(c)g(d).$$

The support of $f \in A_R(\mathcal{G})$, is denoted by

$$\mathrm{supp}(f) = \{b \in \mathcal{G} \mid f(b) \neq 0\} \quad \text{(a clopen subset of } \mathcal{G}\text{)}.$$

Steinberg algebras as skew inverse semigroup rings

Remark

Let $\mathcal{G}$ be a Hausdorff and ample groupoid. The set $\mathcal{G}^a$ of all compact bisections in $\mathcal{G}$ is an *inverse semigroup* under the operations defined by $BC = \{bc \in \mathcal{G} \mid b \in B, c \in C \text{ and } s(b) = r(c)\}$, and $B^* = \{b^{-1} \mid b \in B\}$.

Theorem (Beuter + Gonçalves)

Let $\mathcal{G}$ be a Hausdorff and ample groupoid. Then there is a partial action α of $\mathcal{G}^a$ on $\mathcal{L}_c(\mathcal{G}^{(0)})$ such that the Steinberg algebra $A_R(\mathcal{G})$ is isomorphic, as an R-algebra, to the skew inverse semigroup ring $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$.

Remark

$\mathcal{L}_c(X)$ is the commutative R-algebra of all locally constant, compactly supported, R-valued functions on X (with pointwise addition and multiplication).

Remark

The isomorphism of Theorem 4 is given by the map $\tilde{\psi} : \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a \to A_R(\mathcal{G})$, which is defined on elements of the form $\overline{f_B \delta_B}$, by

$$\tilde{\psi}(\overline{f_B \delta_B})(x) = \begin{cases} f_B(r(x)) & \text{if } x \in B \\ 0 & \text{if } x \notin B, \end{cases}$$

and extended linearly to $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$. In the proof of Theorem 4 it was shown that $\tilde{\psi}$ admits a left inverse, namely the map $\varphi : A_R(\mathcal{G}) \to \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$ defined as follows: Given $f = \sum_{j=1}^n b_j 1_{B_j} \in A_R(\mathcal{G})$, where the B_j's are pairwise disjoint compact bisections of $\mathcal{G}$, let

$$\varphi(f) = \varphi\left(\sum_{i=1}^n b_j 1_{B_j}\right) := \sum_{j=1}^n \overline{b_j 1_{r(B_j)} \delta_{B_j}}.$$

Actually φ is the inverse of $\tilde{\psi}$, and, in particular, it is bijective. By the surjectivity of φ, given any $f \in \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$ we can write

Simplicity of Steinberg algebras

Proposition

Let $\mathcal{G}$ be a Hausdorff and ample groupoid, and let R be a field. Then $\mathcal{G}$ is minimal if, and only if, $\mathcal{L}_c(\mathcal{G}^{(0)})$ is $\mathcal{G}^a$-simple.

Proposition

Let $\mathcal{G}$ be a Hausdorff and ample groupoid. Then $\mathcal{G}$ is effective if, and only if, $\mathcal{D} \simeq \mathcal{L}_c(\mathcal{G}^{(0)})$ is a maximal commutative subring of $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$.

Theorem (Clark et al. & Beuter, Gonçalves, Ö, Royer)

Let $\mathcal{G}$ be a Hausdorff and ample groupoid, and let R be a unital and commutative ring. Then the Steinberg algebra $A_R(\mathcal{G})$ is simple if, and only if, $\mathcal{G}$ is effective, minimal, and R is a field.

The end

THANK YOU FOR YOUR ATTENTION!