Simplicity of associative and non-associative Ore extensions

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The non-associative part is joint work by Patrik Nystedt, Johan Öinert and myself.

Ore extensions, motivation

Introduced by Norwegian mathematician Øystein Ore, under the name of *noncommutative polynomial rings.*

Take a ring R and consider the additive group $R[x]$. Want to give it a new multiplication.

Ore extensions, motivation

Would like $R[x]$ to be an associative ring. Would also like $\deg(ab) = \deg(a) + \deg(b)$ or at least $\deg(ab) \leq \deg(a) + \deg(b)$. Would also like $x^n \cdot x^m = x^{n+m}$.

If $r \in R$ we must have $xr = \sigma(r)x + \delta(r)$, for some functions σ and δ.

In general we must have

$$ax^m \cdot bx^n = \sum_{i \in \mathbb{N}} a\pi_i^m(b)x^{i+n}, \tag{1}$$

for $a, b \in R$ and $m, n \in \mathbb{N}$, where π_i^m denotes the sum of all the $\binom{m}{i}$ possible compositions of i copies of σ and $m - i$ copies of δ in arbitrary order.

Conditions on σ and δ

Want the Ore extension to be a ring.

$$x(r+s) = xr + xs.$$

$$x(rs) = (xr)s.$$

Conditions on σ

σ has to satisfy:

- $\sigma(1) = 1$;
- $\sigma(a+b) = \sigma(a) + \sigma(b)$;
- $\sigma(ab) = \sigma(a)\sigma(b)$.

So σ is an endomorphism.

Conditions on δ

δ must satisfy:

- $\delta(a + b) = \delta(a) + \delta(b)$;
- $\delta(ab) = \sigma(a)\delta(b) + \delta(a)b$.

A δ satisfying this is called a σ-derivation.

A ring

For σ and δ satisfying above conditions we get a ring $R[x;\sigma,\delta]$, called an Ore extension.

Degree

Can measure the *degree* of elements in an Ore extension in the same way as in the polynomial ring. Eg $\deg(x^2 - 3x) = 2$.

$$\deg(ab) = \deg(a) + \deg(b)$$

if σ injective and R does not contain zero-divisors.

Examples

Example

If $\sigma = \mathrm{id}_R$ and $\delta = 0$ then $R[x; \sigma, \delta]$ is isomorphic to $R[x]$, the polynomial ring in one central indeterminate.

Example

If $\sigma = \mathrm{id}_R$ then $R[x; \mathrm{id}_R, \delta]$ is a ring of differential polynomials.

Example

If $\delta = 0$ then $R[x; \sigma, 0]$ is a skew polynomial ring.

Examples II

Example

Take $R = k[y]$, $\sigma(p(y)) = p(qy)$, where $q \in k \setminus \{0, 1\}$ and $\delta(y) = q$. Then $R[x; \sigma, \delta]$ is called the q-Weyl algebra.

Example

Take $R = k[y]$, $\sigma = \mathrm{id}$ and $\delta(y) = 1$. Then $R[x; \sigma, \delta]$ is the ordinary Weyl algebra.

Simple skew polynomial rings

A skew polynomial rings, $R[x; \sigma, 0]$, is never simple since the ideal generated by x is proper.

If δ is a inner derivation, i.e. $\delta(r) = ar - \sigma(r)a$, then $R[x; \sigma, \delta]$ is isomorphic to $R[y; \sigma, 0]$. In particular $R[x; \sigma, \delta]$ is not simple.

Simple Ore extensions with $\sigma \neq \mathrm{id}$

Theorem (Bavula)

Suppose that R is an integral domain, σ is an injective endomorphism and $R[x;\sigma,\delta]$ is a simple ring. Then $\sigma = \mathrm{id}$.

Sketch.

Let k be the field of fractions of R. σ and δ extend to k. Suppose $\sigma(a) \neq a$. For any $b \in R$ we have $\delta(ab) = \delta(ba)$. This gives

$$\sigma(a)\delta(b) + \delta(a)b = \sigma(b)\delta(a) + \delta(b)a \Leftrightarrow (\sigma(a) - a)\delta(b) = \tag{2}$$

$$(\sigma(b) - b)\delta(a) \Leftrightarrow \delta(b) = \frac{\delta(a)}{\sigma(\sigma(a) - a}(\sigma(b) - b). \tag{3}$$

So δ is an inner derivation which is a contradiction. $\square$

Simple Ore extensions with $\sigma \neq \mathrm{id}$ II

Cozzens and Faith construct a simple Ore extension $R[x; \sigma, \delta]$ where R is a division ring and $\sigma \neq \mathrm{id}_R$.

Ideal intersection property for $C_{R[x;\mathrm{id}_R,\delta]}(R)$.

Theorem (Öinert, R., Silvestrov)

If R is a commutative ring then $C_{R[x;\mathrm{id}_R,\delta]}(R)$ has the ideal intersection property. (Meaning it has a non-zero intersection with every non-zero ideal of $R[x;\mathrm{id}_R,\delta]$.)

Proof.

Let I be an ideal in $R[x;\mathrm{id}_R,\delta]$. Take any $a \in I$. If $ar - ra = 0$ for all $r \in R$ we are done. If r is such that $ar - ra \neq 0$ then $ar - ra$ is a non-zero element in I of strictly lower degree than a.By induction we continue this procedure until we obtain a non-zero element contained in $I \cap R'$. If not sooner, this will always occur at degree 0, since $R \subseteq R'$. $\square$

Ideal intersection property for R

Corollary

If R is a maximal commutative subring of $R[x; \mathrm{id}_R, \delta]$ then R has the ideal intersection property.

Necessary condition for simplicity

Theorem

If $R[x; \mathrm{id}_R, \delta]$ is simple then there are no non-trivial δ-invariant ideals of R. (R is said to be δ-simple.) Further δ is an outer derivation.

Proof.

If I is a δ-invariant ideal in R then $I \cdot R[x; \sigma, \delta]$ is an ideal in $R[x; \sigma, \delta]$. The necessity of δ being outer has already been proven. $\square$

Note that for commutative R a non-zero derivation is the same as an outer derivation.

Sufficient conditions for simplicity

Theorem (Öinert, R. and Silvestrov)

Let R be an associative ring. Then $D = R[x; \mathrm{id}_R, \delta]$ is simple if and only if R is δ-simple and $Z(D)$ is a field.

Theorem (Amitsur)

Suppose that R is a simple associative ring and let δ be a derivation on R. If we put $D = R[X; \mathrm{id}_R, \delta]$, then the following assertions hold:

- *Every ideal of D is generated by a unique monic polynomial in $Z(D)$;*
- *There is a monic $b \in R_\delta[X]$, unique up to addition of an element $k \in Z(R)_\delta$, such that $Z(D) = Z(R)_\delta[b]$;*
- *If $\mathrm{char}(R) = 0$ and $b \neq 1$, then there is $c \in R_\delta$ such that $b = c + X$. In that case, $\delta = \delta_c$;*
- *If $\mathrm{char}(R) = p > 0$ and $b \neq 1$, then there is $c \in R_\delta$ and $b_0, \ldots, b_n \in Z(R)_\delta$, with $b_n = 1$, such that $b = c + \sum_{i=0}^{n} b_i X^{p^i}$. In that case, $\sum_{i=0}^{n} b_i \delta^{p^i} = \delta_c$.*

Theorem (Jordan)

Suppose that R is a δ-simple associative ring and let δ be a derivation on R. If we put $D = R[X; \mathrm{id}_R, \delta]$, then the following assertions hold:

(a) *If $\operatorname{char}(R) = 0$, then D is simple if and only if δ is outer;*

(b) *If $\operatorname{char}(R) = p > 0$, then D is simple if and only if no derivation of the form $\sum_{i=0}^{n} b_i \delta^{p^i}$, $b_i \in Z(R)_\delta$, and $b_n = 1$, is an inner derivation induced by an element in R_δ.*

Part II

Non-associative rings

By a non-associative ring we mean a not necessarily associative ring. Must have a unit and must be distributive.

The *center* is the set of all elements that associate and commute with everything. In a simple non-associative ring the center is a field.

Abstract definition

Definition

The pair (S, x) is called a *non-associative Ore extension* of R if the following axioms hold:

(N1) S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(N2) $xR \subseteq R + Rx$;

(N3) $(S, S, x) = (S, x, S) = \{0\}$.

If (N2) is replaced by

(N2)′ $[x, R] \subseteq R$;

then (S, x) is called a *non-associative differential polynomial ring over* R.

Construction

Let σ and δ be additive maps such that $\sigma(1) = 1$ and $\delta(1) = 0$. As before we equip $R[X]$ with a new multiplication.

The ring structure on $R[X; \sigma, \delta]$ is defined on monomials by

$$aX^m \cdot bX^n = \sum_{i \in \mathbb{N}} a\pi_i^m(b)X^{i+n}, \tag{4}$$

for $a, b \in R$ and $m, n \in \mathbb{N}$, where π_i^m denotes the sum of all the $\binom{m}{i}$ possible compositions of i copies of σ and $m - i$ copies of δ in arbitrary order.

Definition

Suppose that (S, x) is a non-associative Ore extension of R. Put $R_x = \{a \in R \mid ax = xa\}$. We say that (S, x) is *strong* if at least one of the following axioms holds:

(N4) $(x, R, R_x) = \{0\}$;

(N5) $(x, R_x, R) = \{0\}$.

In that case we call R_x *the ring of constants of* R.

Set $R_\delta^\sigma = \{r | \sigma(r) = r,\ \delta(r) = 0\}$.

Theorem

Every non-associative Ore extension of R is isomorphic to a generalized polynomial ring $R[X;\sigma,\delta]$. If the non-associative Ore extension is strong, then σ and δ are both right R_δ^σ-linear or both are R_δ^σ-linear.

It is easy to see that R_δ^σ is the ring of constants.

Theorem

Suppose that R is a non-associative ring and that δ right or left linear over the constants. If we put $D = R[X; \mathrm{id}_R, \delta]$, then the following assertions hold:

(a) *If R is δ-simple, then every ideal of D is generated by a unique monic polynomial in $Z(D)$;*

(b) *If R is δ-simple, then there is a monic $b \in R_\delta[X]$, unique up to addition of an element $k \in Z(R)_\delta$, such that $Z(D) = Z(R)_\delta[b]$;*

(c) *D is simple if and only if R is δ-simple and $Z(D)$ is a field. In that case $Z(D) = Z(R)_\delta$ in which case $b = 1$;*

(d) *If R is δ-simple, δ is a derivation on R and $\mathrm{char}(R) = 0$, then either $b = 1$ or there is $c \in R_\delta$ such that $b = c + X$. In the latter case, $\delta = \delta_c$;*

(e) *If R is δ-simple, δ is a derivation on R and $\mathrm{char}(R) = p > 0$, then either $b = 1$ or there is $c \in R_\delta$ and $b_0, \ldots, b_n \in Z(R)_\delta$, with $b_n = 1$, such that $b = c + \sum_{i=0}^{n} b_i X^{p^i}$. In the latter case, $\sum_{i=0}^{n} b_i \delta^{p^i} = \delta_c$.*

Associative coefficients

Theorem

Suppose that $D = R[X; \mathrm{id}_R, \delta]$ is a non-associative differential polynomial ring such that R is associative and all positive integers are regular in R. If R is δ-simple but δ is not a derivation, then D is simple.

References

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