Undeformed commutators in $q$-deformed Heisenberg algebras

Rafael Cantuba

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Definition

Let $\mathbb{F}$ be a field, and let $q \in \mathbb{F}$. The $q$-deformed Heisenberg algebra $\mathcal{H}(q)$ is the unital associative algebra over $\mathbb{F}$ that has a presentation by generators $A, B$ and relation

$$AB - qBA = I,$$

where $I$ is the multiplicative identity.
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A Lie algebra structure is induced by $[U, V] := UV - VU$ for all $U, V \in \mathcal{H}(q)$. 

Goal: Study the Lie subalgebra $L(q)$ of $\mathcal{H}(q)$ generated by $A, B$. 

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Some notation:

\[ F = \text{arbitrary field,} \]
\[ N = \text{set of all nonnegative integers,} \]
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Some preliminary notions:

1. Let \( t \in \mathbb{N} \). By a word of length \( t \) on \( X \) we mean a finite sequence of the form

\[ X_1 X_2 \cdots X_t \]

where \( X_i \in \mathcal{X} \) for all \( i \in \{1, 2, \ldots, t\} \).
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4. Given words $X_1 X_2 \cdots X_s$ and $Y_1 Y_2 \cdots Y_t$ on $\mathcal{X}$, their *concatenation product* is the word

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2. The multiplication operation in \( \mathbb{F}\langle \mathcal{X}\rangle \) is completely determined by the concatenation product of words on \( \mathcal{X} \).
3. Let \( f_1, f_2, \ldots, f_k \in \mathbb{F}\langle \mathcal{X}\rangle \), and let \( \mathcal{J} \) be the (two-sided) ideal of \( \mathbb{F}\langle \mathcal{X}\rangle \) generated by \( f_1, f_2, \ldots, f_k \). Denote the elements of \( \mathcal{X} \) by \( G_1, G_2, \ldots, G_n \). Then the algebra defined by a presentation having generators \( G_1, G_2, \ldots, G_n \) and relations \( f_1 = 0, f_2 = 0, \ldots, f_k = 0 \) is precisely the quotient algebra \( \mathbb{F}\langle \mathcal{X}\rangle \slash \mathcal{J} \).
Example

Let $q \in \mathbb{F}$, and set $\mathcal{X} = \{A, B\}$. Denote by $\mathcal{J}$ the ideal of $\mathbb{F} \langle \mathcal{X} \rangle$ generated by $AB - qBA - I$. Then $\mathcal{H}(q) = \mathbb{F} \langle \mathcal{X} \rangle / \mathcal{J}$. 
The free Lie algebra on $\mathcal{X}$ (or the set of all Lie polynomials in $\mathcal{X}$) is the Lie subalgebra $\mathcal{L} := \mathcal{L}_\mathcal{X}$ of $\mathbb{F} \langle \mathcal{X} \rangle$ generated by $\mathcal{X}$.
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2. Analogous to a property of $\mathbb{F}\langle \mathcal{X} \rangle$: Any Lie algebra generated by $|\mathcal{X}|$ elements is a quotient of $\mathcal{L}$.
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3. Given an ideal $\mathcal{I}$ of $\mathbb{F}\langle \mathcal{X} \rangle$, the Lie subalgebra of $\mathbb{F}\langle \mathcal{X} \rangle / \mathcal{I}$ generated by
The free algebra $F\langle X \rangle$ and the free Lie algebra on $X$

1. The free Lie algebra on $X$ (or the set of all Lie polynomials in $X$) is the Lie subalgebra $L := L_X$ of $F\langle X \rangle$ generated by $X$.

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The free Lie algebra on $\mathcal{X}$ (or the set of all Lie polynomials in $\mathcal{X}$) is the Lie subalgebra $\mathcal{L} := \mathcal{L}_\mathcal{X}$ of $\mathbb{F}\langle \mathcal{X} \rangle$ generated by $\mathcal{X}$.

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Preliminaries

We are interested in Lie algebras related to $\mathbb{F}\langle \mathcal{X} \rangle$ described in the following.

1. **The free Lie algebra on $\mathcal{X}$** . . .
2. **. . . the Lie algebra over $\mathbb{F}$ . . . [with] . . . generators $G_1, G_2, \ldots G_n$ and relations $f_1 = 0, f_2 = 0, \ldots, f_k = 0$ . . .**
3. **Given an ideal $\mathcal{J}$ of $\mathbb{F}\langle \mathcal{X} \rangle$, the Lie subalgebra of $\mathbb{F}\langle \mathcal{X} \rangle / \mathcal{J}$ generated by $\mathcal{X}$ (or the set of all Lie polynomials in $\mathcal{X}$ in the algebra $\mathbb{F}\langle \mathcal{X} \rangle / \mathcal{J}$) is precisely $\mathcal{L}/(\mathcal{J} \cap \mathcal{L})$.

**Proposition**

*With reference to above notation, given the canonical map $\varphi : \mathbb{F}\langle \mathcal{X} \rangle \to \mathbb{F}\langle \mathcal{X} \rangle / \mathcal{J}$, and a basis $\mathcal{B}$ of $\mathcal{L}$ then a spanning set for the Lie algebra $\mathcal{L}/(\mathcal{J} \cap \mathcal{L})$ consists of vectors of the form $

\varphi(X), \quad (X \in \mathcal{B}).
Proposition

With reference to above notation, given the canonical map \( \varphi : \mathbb{F}\langle X \rangle \rightarrow \mathbb{F}\langle X \rangle / J \), and a basis \( B \) of \( L \) then a spanning set for the Lie algebra \( L/(J \cap L) \) consists of vectors of the form

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Example

Let \( q \in \mathbb{F} \), and set \( X = \{ A, B \} \). Denote by \( J \) the ideal of \( \mathbb{F}\langle X \rangle \) generated by \( AB - qBA - I \). Then \( \mathcal{H}(q) = \mathbb{F}\langle X \rangle / J \).
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Example

Let \( q \in \mathbb{F} \), and set \( X = \{A, B\} \). Denote by \( J \) the ideal of \( \mathbb{F}\langle X \rangle \) generated by \( AB - qBA - I \). Then \( H(q) = \mathbb{F}\langle X \rangle / J \). The object of our study is the Lie subalgebra of \( H(q) \) generated by \( A, B \) which is precisely

\[ \mathcal{L}(q) := L/(J \cap L). \]
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Let \( q \in \mathbb{F} \), and set \( \mathcal{X} = \{A, B\} \). Denote by \( \mathcal{J} \) the ideal of \( \mathbb{F} \langle \mathcal{X} \rangle \) generated by \( AB - qBA - I \). Then \( \mathcal{H}(q) = \mathbb{F} \langle \mathcal{X} \rangle / \mathcal{J} \). The object of our study is the Lie subalgebra of \( \mathcal{H}(q) \) generated by \( A, B \) which is precisely

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From this point onward, we fix $\mathcal{X} = \{A, B\}$. 

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3. Given a basis for $\mathcal{L}(q)$, compute the commutator table.
A basis for $\mathcal{L}$ consisting of regular words on $A, B$

The following notions are from the formulation given in (Ufnarovskij, 1995). For a full discussion on regular words on $n$ generators, refer to arXiv:1709.02612 (Section 2).
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**Definition**

Fix the ordering $A < B$ on $\mathcal{X}$.
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**Definition**

Fix the ordering $A < B$ on $X$. Denote two arbitrary distinct nonempty words on $A, B$ by

$$U = X_1 X_2 \cdots X_{|U|},$$
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Rafael Cantuba

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**Definition**

A word on $A, B$ is regular if it is a generator or if, with respect to $\triangleleft$, it is strictly greater than any of its cyclic permutations.
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**Definition**

A word on $A, B$ is regular if it is a generator or if, with respect to $\prec$, it is strictly greater than any of its cyclic permutations.

Example 1

The words $B^3AB^2A$ and $B^3A$ are regular.

Example 2

The words $B^2AB^3A$, $B^2$, and $A^3$ are not.

Lemma

If the word $W$ is regular, and if $V$ is the length-maximal proper ending of $W$ that is also regular, and if $U$ is the word such that $W = UV$, then $U$ is regular.
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*If the word $W$ is regular, and if $V$ is the length-maximal proper ending of $W$ that is also regular, and if $U$ is the word such that $W = UV,*
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Definition

We define $[A] := A$, and $[B] := B$ as the regular nonassociative words on $A, B$ of length 1.
Lemma

If the word $W$ is regular, and if $V$ is the length-maximal proper ending of $W$ that is also regular, and if $U$ is the word such that $W = UV$, then $U$ is regular.

Definition

We define $[A] := A$, and $[B] := B$ as the regular nonassociative words on $A, B$ of length 1. Given $t \in \mathbb{Z}^+$, suppose that all regular nonassociative words on $A, B$ of lengths strictly less than $t$ have been defined.
A basis for $\mathcal{L}$ consisting of regular words on $A, B$

**Lemma**

*If the word $W$ is regular, and if $V$ is the length-maximal proper ending of $W$ that is also regular, and if $U$ is the word such that $W = UV$, then $U$ is regular.*

**Definition**

We define $[A] := A$, and $[B] := B$ as the *regular nonassociative words* on $A, B$ of length 1. Given $t \in \mathbb{Z}^+$, suppose that all regular nonassociative words on $A, B$ of lengths strictly less than $t$ have been defined. Then given a regular word $W$ of length $t$ expressible as $W = UV$ according to the above lemma, we define $[W] := [[U], [V]].$
A basis for $\mathcal{L}$ consisting of regular words on $A, B$

**Lemma**

If the word $W$ is regular, and if $V$ is the length-maximal proper ending of $W$ that is also regular, and if $U$ is the word such that $W = UV$, then $U$ is regular.

**Definition**

We define $[A] := A$, and $[B] := B$ as the regular nonassociative words on $A, B$ of length 1. Given $t \in \mathbb{Z}^+$, suppose that all regular nonassociative words on $A, B$ of lengths strictly less than $t$ have been defined. Then given a regular word $W$ of length $t$ expressible as $W = UV$ according to the above lemma, we define $[W] := [[U], [V]]$.

**Theorem**

The regular nonassociative words on $A, B$ form a basis for the free Lie algebra on $A, B$. 

Rafael Cantuba

Undeformed commutators in $q$-deformed Heisenberg algebras
A basis for \( \mathcal{L} \) consisting of regular words on \( A, B \)

**Theorem**

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**Example**

1. \( \left[ BA^4 \right] = \)
A basis for $\mathcal{L}$ consisting of regular words on $A, B$

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A basis for $\mathcal{L}$ consisting of regular words on $A, B$

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Rafael Cantuba

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5. $[B^3A^4BA] = [B, [[[B^2A^4], [BA]]] = \cdots$
6. $[BAB^3A^4]$ is undefined.
Effect of the relation $AB - qBA - I = 0$ on the regular words

To know which of the regular nonassociative words on $A, B$ (spanning set elements of $\mathcal{L}(q)$) can be removed and obtain a maximal linearly independent set, the following result was consequential.

Lemma (Hellström and Silvestrov, 2005)
The following vectors form a basis for $H^q$.

\[
\begin{align*}
[A, B]^k, & \quad [A, B]^k A_l, B_l [A, B]^k, \\
(k \in \mathbb{N}, l \in \mathbb{Z}^+) & \quad (1)
\end{align*}
\]

How is the product of any two basis elements in (1) expressible as a linear combination of (1)?
Effect of the relation $AB - qBA - I = 0$ on the regular words

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Effect of the relation $AB - qBA - I = 0$ on the regular words

How is the product of any two basis elements in $[A, B]^k, [A, B]^k A^l, B^l [A, B]^k, (k \in \mathbb{N}, l \in \mathbb{Z}^+)$. expressible as a linear combination of such vectors? Towards answer: the following consequences of the simple relation $AB - qBA = I$. 

Rafael Cantuba

Undeformed commutators in $q$-deformed Heisenberg algebras
Effect of the relation $AB - qBA - I = 0$ on the regular words

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$$[A, B]^k B^l = q^{kl} B^l [A, B]^k,$$

$$A^l [A, B]^k = q^{kl} [A, B]^k A^l,$$

$$B^l A^l = q^{-\binom{l}{2}} (q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{l-i}{2}} \binom{l}{i} q [A, B]^i,$$

$$A^l B^l = (q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{i+1}{2}} \binom{l}{i} q [A, B]^i.$$
Effect of the relation $AB - qBA - I = 0$ on the regular words

Some consequences of the simple relation $AB - qBA = I$.

$$[A, B]^k B^l = q^{kl} B^l [A, B]^k,$$
$$A^l [A, B]^k = q^{kl} [A, B]^k A^l,$$
$$B^l A^l = q^{-\binom{l}{2}} (q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{i}{2}} \frac{\{l\}}{\{i\}_q} [A, B]^i,$$
$$A^l B^l = (q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{i+1}{2}} \frac{\{l\}}{\{i\}_q} [A, B]^i,$$

where the expression $\frac{\{l\}}{\{i\}_q}$ is as described in the following: Given $n \in \mathbb{N}$, let
$$\{n\}_q := 1 + q + q^2 + \cdots + q^{n-1},$$
and $$\{n\}_q! := \{n\}_q \{n-1\}_q \cdots \{1\}_q.$$ If $k \in \mathbb{N}$ with $k \leq n$, we define the number $\frac{\{n\}_q!}{\{k\}_q! \{n-k\}_q!}$ as 1 if $k \in \{0, n\}$, or as the expression $$\frac{\{n\}_q!}{\{k\}_q! \{n-k\}_q!},$$ otherwise.
Effect of the relation $AB - qBA - I = 0$ on the regular words

Some consequences of the simple relation $AB - qBA = I$.

\begin{align*}
[A, B]^k B^l &= q^{kl} B^l [A, B]^k, \quad (2) \\
A^l [A, B]^k &= q^{kl} [A, B]^k A^l, \quad (3) \\
B^l A^l &= q^{-\binom{l}{2}}(q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{l-i}{2}} \binom{l}{i}_q [A, B]^i, \quad (4) \\
A^l B^l &= (q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\binom{i+1}{2}} \binom{l}{i}_q [A, B]^i, \quad (5)
\end{align*}

Relations (2) to (4) are also from (Hellström and Silvestrov, 2005), while (5) was proven using routine computations and arguments (arXiv:1709.02612, Proposition 3.3).
Effect of the relation $AB - qBA - I = 0$ on the regular words

Some consequences of the simple relation $AB - qBA = I$.

\[ [A, B]^k B^l = q^{kl} B^l [A, B]^k , \tag{2} \]
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\[ B^l A^l = q^{-\left(\frac{l}{2}\right)} (q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\left(\frac{l-i}{2}\right)} \left(\begin{array}{c} l \\ i \end{array}\right)_q [A, B]^i , \tag{4} \]
\[ A^l B^l = (q - 1)^{-l} \sum_{i=0}^{l} (-1)^{l-i} q^{\left(\frac{i+1}{2}\right)} \left(\begin{array}{c} l \\ i \end{array}\right)_q [A, B]^i , \tag{5} \]

Relations (2) to (4) are also from (Hellström and Silvestrov, 2005), while (5) was proven using routine computations and arguments (arXiv:1709.02612, Proposition 3.3). These relations were of significance in the proof of:
Effect of the relation $AB - qBA - I = 0$ on the regular words

Theorem (Cantuba, 2017)

If $q$ is nonzero and is not a root of unity, then the following vectors form a basis for $\mathcal{L}(q)$.

$$ A, B, [BA], \left[(BA)^k BA^{l+1}\right], \left[B^{l+1} A(BA)^k\right], \left[B(BA)^k BA^2\right], \quad (k \in \mathbb{N}, l \in \mathbb{Z}^+) $$
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If $q$ is nonzero and is not a root of unity, then the following vectors form a basis for $\mathcal{L}(q)$.

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$$\biggl[ B^{l+1} A(BA)^k \biggr], \biggl[ B(BA)^k BA^2 \biggr], \quad (k \in \mathbb{N}, l \in \mathbb{Z}^+).$$

$$A, B, [A, B]^k, \quad [A, B]^k A^l, \quad B^l [A, B]^k, \quad (k, l \in \mathbb{Z}^+).$$
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(details of the commutator table in arXiv 1709.02612, Section 5)
Other properties of $\mathcal{L}(q)$

Proposition (Cantuba, 2017)

$\mathcal{H}(q) = \mathcal{L}(q) \oplus \text{Span} \{I, A^2, B^2, A^3, B^3, \ldots\}$. 

Corollary (Cantuba, 2017)
The Lie algebra $\mathcal{L}(q)$ is a Lie ideal of $\mathcal{H}(q)$. The resulting quotient Lie algebra has an infinite basis consisting of mutually commuting elements.
Other properties of $\mathcal{L}(q)$

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The Lie algebra $\mathcal{L}(q)$ is a Lie ideal of $\mathcal{H}(q)$. The resulting quotient Lie algebra as an infinite basis consisting of mutually commuting elements.
For the case $\mathbb{F} = \mathbb{C}$, and $q \in ]0, 1[$:

Using a result from (Hellström and Silvestrov, 2005), $H(q)$ is faithfully represented by Hilbert space operators on the sequence space $\ell_2(N)$.

The generator $B$ is represented by a unilateral weighted shift.

The generator $A$ performs the role of the adjoint of $B$, (adjoint in the Hilbert space sense, and not in the context of Lie algebra derivations).

The element $[A, B]$ is hence a diagonal operator and is Hermitian.

The ideal of all the compact operators in $H(q)$ is precisely the derived (Lie) algebra of $L(q)$.

The resulting Calkin algebra is the complex Laurent polynomial algebra in one indeterminate.
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Detour into Operator Theory

For the case $\mathbb{F} = \mathbb{C}$, and $q \in ]0, 1[$:

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6. The resulting Calkin algebra is the complex Laurent polynomial algebra in one indeterminate.
The original setting for this type of Lie algebra problem

The Fairlie-Odesskii algebra:
\( U'_q(\mathfrak{so}_3) := \) the algebra with generators \( l_1, l_2, l_3 \) and relations

\[
\begin{align*}
q^{\frac{1}{2}} l_1 l_2 - q^{-\frac{1}{2}} l_2 l_1 &= l_3, \\
q^{\frac{1}{2}} l_2 l_3 - q^{-\frac{1}{2}} l_3 l_2 &= l_1, \\
q^{\frac{1}{2}} l_3 l_1 - q^{-\frac{1}{2}} l_1 l_3 &= l_2.
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The universal Askey-Wilson algebra:

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The universal Askey-Wilson algebra:
some form of generalization encompassing \( U'_q(\mathfrak{so}_3) \) and some other algebras introduced in (Terwilliger, 2011)

\[
\vdots
\]

(same type of Lie algebra problems for the above algebras)


Thank you for your attention!!!