Geometric aspects of noncommutative principal bundles

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1. Noncommutative principal bundles

**Convention**
Throughout this talk $G$ is assumed to be a compact group and $A$ is assumed to be a unital $\mathcal{C}^*$-algebra.

We call a group homomorphism $\alpha : G \to \text{Aut}(A)$ an action of $G$ on $A$ if for each $x \in A$ the map $g \mapsto \alpha_g(x)$ is continuous.

**Definition (Free actions on $\mathcal{C}^*$-algebras)**
An action $\alpha : G \to \text{Aut}(A)$ is called free if the Ellwood map $\Phi : A \otimes \text{alg} A \to \mathcal{C}(G, A)$, $\Phi(x \otimes y)(g) := x \alpha_g(y)$ has dense range.
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Definition (Free actions on $C^*$-algebras)

An action $\alpha : G \rightarrow \text{Aut}(\mathcal{A})$ is called *free* if the *Ellwood map*

$$\Phi : \mathcal{A} \otimes_{\text{alg}} \mathcal{A} \rightarrow C(G, \mathcal{A}), \quad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range.
Example (Classical actions)

Let $\sigma : X \times G \to X$ be a continuous action of $G$ on a compact space $X$. Then the map $\alpha : G \to \text{Aut}(C(X))$ defined by $\alpha_g(f)(x) := f(\sigma(x, g))$ is an action of $G$ on $C(X)$. Moreover, the following statements are equivalent:

(a) The map $\sigma$ is free in the classical sense, i.e., all stabilizers are trivial.

(b) The map $X \times G \to X \times X$, $(x, g) \mapsto (x, \sigma(x, g))$ is injective.

(c) The action $\alpha : G \to \text{Aut}(C(X))$ is free in the sense above.

Remark (Smooth principal bundles)

In the smooth category there is a bijective correspondence between free (and proper) group actions and locally trivial principal bundles.
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Remark (Smooth principal bundles)

In the smooth category there is a bijective correspondence between free (and proper) group actions and \textit{locally trivial principal bundles}. 
Example (Quantum 2-tori)

Let $\theta \in \mathbb{R}$. 

The quantum $2$-torus $T^2_{\theta}$ is the universal $C^*$-algebra generated by unitaries $U$, $V$ satisfying the commutation relation $UV = \exp(2\pi i \theta) VU$.

We point out that $T^2_{\theta} \cong C(T^2) \equiv T^2$ if and only if $\theta \in \mathbb{Z}$.

Moreover, the map $\alpha: T^2 \to \text{Aut}(T^2_{\theta})$ given on generators by $\alpha(z, w)(U) := z \cdot U$ and $\alpha(z, w)(V) := w \cdot V$ is a free and ergodic action of $T^2$ on $T^2_{\theta}$.
Example (Quantum 2-tori)

Let \( \theta \in \mathbb{R} \). The **quantum 2-torus** \( \mathbb{T}_\theta^2 \) is the universal C*-algebra generated by unitaries \( U, V \) satisfying the commutation relation

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We point out that $\mathbb{T}_\theta^2 \cong C(\mathbb{T}^2)$ (\(\triangleq\) $\mathbb{T}^2$) if and only if $\theta \in \mathbb{Z}$. Moreover, the map $\alpha : \mathbb{T}^2 \to \text{Aut}(\mathbb{T}_\theta^2)$ given on generators by

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Let $q \in [-1, 1]$. Woronowicz’s quantum $SU_q(2)$ is the universal $C^*$-algebra generated by two elements $a$ and $c$ subject to the five relations

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    a^*a + cc^* = 1, \quad aa^* + q^2cc^* = 1, \quad cc^* = c^*c, \\
    ac = qca, \quad \text{and} \quad ac^* = qc^*a.
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The map \( \alpha : \mathbb{T} \to \text{Aut}(SU_q(2)) \) given on generators by

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is a free action of $\mathbb{T}$ on $SU_q(2)$. It is called the quantum Hopf fibration.
Why studying free actions?

The investigation and classification of actions of (quantum-) groups on $C^*$-algebras is intrinsically interesting. Free actions are closely related to the theory of Hopf-Galois extensions and the study of strongly graded rings. Noncommutative principal bundles are becoming increasingly prevalent in applications to topology, geometry, and mathematical physics:

- They appear in the study of 3-dim TQFT's that are based on the modular tensor category of representations of the Drinfeld double.
- Their applications in T-duality may lead to a better understanding of T-duals and the question of their existence.
- They may be used to develop a theory of quantum gerbes and a fundamental group for noncommutative spaces (cf. [1]).
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2. Classification of noncommutative principal bundles

Remark (Classification of smooth principal bundles)
Given a smooth manifold $M$ and a Lie group $G$, Čech cohomology provides a method for classifying smooth principal $G$-bundles over $M$. In fact, $\text{PBUN}(M, G) \cong \check{\text{H}}^1(M, G)$.

We recall that smooth principal bundles correspond bijectively to free (and proper) group actions. We may therefore approach a possible classification of noncommutative principal bundles in the following way:

Problem (Classification of free actions)
Given a unital $C^*$-algebra $B(\hat{=\mathbb{C}(M)})$ and a compact group $G$, understand and classify all free actions $\alpha : G \to \text{Aut}(A)$ such that $A_G = B$.
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Problem (Classification of free actions)

Given a unital C*-algebra $B (\cong C(M))$ and a compact group $G$, understand and classify all free actions $\alpha : G \to \text{Aut}(\mathcal{A})$ such that $\mathcal{A}^G = B$. 
Remark (Structure theory of actions)

Let $\alpha : G \rightarrow \text{Aut}(A)$ be any action with fixed point algebra $B$. As every representation of $G$, the algebra $A$ can be decomposed into its isotypic components $A(\pi), \pi \in \hat{G}$, and $\bigoplus_{\pi \in \hat{G}} A(\pi)$ is dense in $A$.

Each $A(\pi), \pi \in \hat{G}$ carries a natural Hilbert $B$-module structure w.r.t. $\langle x, y \rangle_B := P_0(x^*y) := \int_G \alpha_g(x^*y) \, dg, x, y \in A(\pi)$.

The multiplication between isotypic components is well captured by family of maps (fusion rules) $m_{\pi,\rho} : A(\pi) \otimes_B A(\rho) \rightarrow A(\pi \otimes \rho)$, $m_{\pi,\rho}(x \otimes y) := x \cdot y$.

For free actions the fusion rules are particularly good-natured which makes classification certainly (more) available.
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- For free actions the fusion rules are particularly good-natured which makes classification certainly (more) available.
Remark (Factor systems of free actions)

Let $\alpha : G \to \text{Aut}(A)$ be a free action with fixed point algebra $B$.

For each representation $(\pi, V_\pi)$ of $G$ there is a Hilbert space $H_\pi$ and a coisometry $s(\pi) \in L(H_\pi, V_\pi \otimes A)$ satisfying

$$\alpha_g(s(\pi)) = \pi^* g s(\pi) \quad \forall g \in G.$$ 

For each representation $\pi$ of $G$ we define the $^*$-homomorphism $\gamma_\pi : B \to L(H_\pi \otimes B)$,

$$\gamma_\pi(b) = s(\pi)^* (1_{V_\pi \otimes b}) s(\pi).$$

and for each pair $\pi, \rho$ of representations of $G$ an element $\omega(\pi, \rho) := s(\pi \otimes \rho)^* s(\pi) s(\rho) \in L(H_\pi \otimes H_\rho, H_\pi \otimes H_\rho \otimes B)$.

The corresponding collection $(H_\pi, \gamma, \omega) = (H_\pi, \gamma_\pi, \omega(\pi, \rho))_{\pi, \rho} \in \hat{G}$ is called a factor system of $\alpha : G \to \text{Aut}(A)$. 

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- The corresponding collection $(\mathcal{H}, \gamma, \omega) = (\mathcal{H}_\pi, \gamma_\pi, \omega(\pi, \rho))_{\pi, \rho \in \hat{G}}$ is called a factor system of $\alpha : G \to \text{Aut}(A)$. 
Factor systems are the key feature in our research program. In fact, they satisfy interesting algebraic relations that make free actions accessible to classification, $K$-theoretic considerations, and computations in general.
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**Theorem (Schwieger-W. 15’,16’,17’)**

Let $\mathcal{B}$ be a unital $C^*$-algebra and $G$ a compact group. In [2–4] we provided a complete classification of free actions of $G$ with fixed point algebra $\mathcal{B}$ in terms of *Hilbert $\mathcal{B}$-modules* and *factor systems*.
3. Geometric aspects of noncommutative principal bundles

Many important geometric concepts like connections, parallel transport, curvature, and characteristic classes depend on the choice of a connection 1-form (a.k.a. geometric distribution) on some principal bundle.

Remark (The Athiya sequence)
Given a principal $G$-bundle $\pi: P \rightarrow M$, connection 1-forms on $P$ are in a $1:1$-correspondence with $C^\infty(M)$-linear sections of the Athiya-Sequence $0 \rightarrow \text{gau}(P) \rightarrow V(P) \rightarrow G \rightarrow V(M) \rightarrow 0$.

Heuristic noncommutative approach:
Given a free action $\alpha: G \rightarrow \text{Aut}(A)$ with fixed point algebra $B$, study its geometric aspects in terms of a "generalized Athiya sequence" $\text{der}_G(A) \rightarrow \text{der}(B), \delta \mapsto \delta|_B$. 

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Heuristic noncommutative approach: Given a free action $\alpha: G \to \text{Aut}(A)$ with fixed point algebra $B$, study its geometric aspects in terms of a \"generalized Athiya sequence\" $\mathfrak{der} G(A) \to \mathfrak{der}(B)$, $\delta \mapsto \delta|_B$. 
3. Geometric aspects of noncommutative principal bundles

Many important geometric concepts like connections, parallel transport, curvature, and characteristic classes depend on the choice of a connection 1-form (a.k.a. geometric distribution) on some principal bundle.

**Remark (The Athiya sequence)**

Given a principal $G$-bundle $q : P \rightarrow M$, connection 1-forms on $P$ are in a $1:1$-correspondence with $C^\infty(M)$-linear sections of the Athiya-Sequence

$$0 \rightarrow \text{gau}(P) \rightarrow \mathcal{V}(P)^G \rightarrow \mathcal{V}(M) \rightarrow 0.$$
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Heuristic noncommutative approach:

Given a free action $\alpha : G \to \text{Aut}(A)$ with fixed point algebra $B$, study its geometric aspects in terms of a “generalized Athiya sequence”

$$\text{der}_G(A) \to \text{der}(B), \quad \delta \mapsto \delta|_B.$$
The main challenge of this approach is to find suitable conditions that help to decide whether a given \( \ast \)-derivation on \( \mathcal{B} \) extends to a \( \mathcal{G} \)-equivariant \( \ast \)-derivation on \( \mathcal{A} \).

Statement of the main problem:

Let \( \alpha : \mathcal{G} \to \text{Aut}(\mathcal{A}) \) be a free action with fixed point algebra \( \mathcal{B} \) and \( \mathcal{B}_0 (\hat{=} C^\infty(\mathcal{M})) \) a dense unital \( \ast \)-subalgebra of \( \mathcal{B} \).

Is there (i) a \( \mathcal{G} \)-invariant, dense unital \( \ast \)-subalgebra \( \mathcal{A}_0 (\hat{=} C^\infty(\mathcal{P})) \) of \( \mathcal{A} \) with \( \mathcal{A}_0 \cap \mathcal{B} = \mathcal{B}_0 \), and (ii) a way to extend a given \( \ast \)-derivation \( \delta_{|\mathcal{B}_0} : \mathcal{B}_0 \to \mathcal{B}_0 \) to a \( \mathcal{G} \)-equivariant \( \ast \)-derivation \( \delta_{|\mathcal{A}_0} : \mathcal{A}_0 \to \mathcal{A}_0 \).
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(ii) a way to extend a given $\ast$-derivation $\delta_B : B_0 \to B_0$ to a $G$-equivariant $\ast$-derivation $\delta_A : A_0 \to A_0$. 
Why is this interesting?

Geometric aspects of noncommutative principal bundles have not been studied yet in a conclusive way, mainly due to the absence of a simple notion of a "differentiable structure". Our results may be used to transfer the notions of connection 1-forms, connections, parallel transport, curvature, and characteristic classes to the noncommutative setting. The mathematical description for classical gauge theories is given in terms of smooth principal bundles. Hence, our analysis could yield a natural framework for studying noncommutative gauge theories.
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**Definition (Cleft actions)**

An action $\alpha: G \to \text{Aut}(A)$ is called cleft if there exists a unitary element $u \in M(G) \otimes A$ satisfying $\alpha_g(u) = \lambda_g^* u$ for all $g \in G$.

**Remark (Cleft actions)**

The quantum torus $T^2_\theta$ together with its canonical free and ergodic $T^2$-action $\alpha: T^2 \to \text{Aut}(T^2_\theta)$ as described before is cleft. Each cleft action is free, but the converse does not hold. For instance, the quantum Hopf fibration is not cleft. Cleft means that the coisometries discussed before are in fact unitaries and the element $u \in M(G) \otimes A$ is just the collection of all $u_\pi, \pi \in \hat{G}$. 

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**Theorem (W. 18')**

Let $\mathcal{B}_0$ be a dense unital $*$-subalgebra of $\mathcal{B}$. Moreover, let $\delta_\mathcal{B} : \mathcal{B}_0 \to \mathcal{B}_0$ be a $*$-derivation. Then the following assertions hold:
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**Theorem (W. 18’)**

Let $B_0$ be a dense unital $*$-subalgebra of $B$. Moreover, let $\delta_B : B_0 \to B_0$ be a $*$-derivation. Then the following assertions hold:

(a) The set

$$A_0 := \{ \text{Tr}(ux) \mid x \in M_0(G) \otimes B_0 \}$$

gives a $G$-invariant, dense unital $*$-subalgebra of $A$ with $A_0 \cap B = B_0$. 

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Theorem continued (W. 18’)

(b) If $\delta_A : A_0 \to A_0$ is an $G$-equivariant $*$-derivation that extends $\delta_B$, then $H := -\hat{\omega}^* \delta_A(u) \in M(G) \otimes B_0$ is self-adjoint and satisfies

$$\gamma \delta_B(b) - \delta_B \gamma(b) = \hat{\omega}[H, \gamma(b)], \quad \forall b \in B_0,$$

and

$$-\hat{\omega}^* \delta_B(\omega) = id \otimes \gamma(H) + H^2 - \omega^* \Delta(H) \omega.$$
(b) If $\delta_A : \mathcal{A}_0 \to \mathcal{A}_0$ is an $G$-equivariant $\ast$-derivation that extends $\delta_B$, then $H := -\iota u^* \delta_A(u) \in M(G) \otimes \mathcal{B}_0$ is self-adjoint and satisfies

$$\gamma \delta_B(b) - \delta_B \gamma(b) = \iota [H, \gamma(b)], \quad \forall b \in \mathcal{B}_0, \quad (1)$$

$$-\iota \omega^* \delta_B(\omega) = \text{id} \otimes \gamma(H) + H_2 - \omega^* \Delta(H)\omega. \quad (2)$$

(c) If $H \in M(G) \otimes \mathcal{B}_0$ is self-adjoint and satisfies (1) and (2), then

$$\delta_A(\text{Tr}(ux)) := \text{Tr}(u \delta_B(x) + \iota u H x), \quad x \in M_0(G) \otimes \mathcal{B}_0,$$

is a well-defined $G$-equivariant $\ast$-derivation that extends $\delta_B$. 
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(b) If $\delta_A : A_0 \to A_0$ is an $G$-equivariant $^*$-derivation that extends $\delta_B$, then $H := -u^* \delta_A(u) \in M(G) \otimes B_0$ is self-adjoint and satisfies

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(b) If $\delta_A : A_0 \to A_0$ is an $G$-equivariant $*$-derivation that extends $\delta_B$, then $H := -\nu u^* \delta_A(u) \in M(G) \otimes B_0$ is self-adjoint and satisfies

\[
\begin{align*}
\gamma\delta_B(b) - \delta_B\gamma(b) &= \nu[H, \gamma(b)], \quad \forall b \in B_0, \\
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- Study interesting classes of examples such as Woronowicz’s $SU_q(2)$ and the Connes-Landi spheres
- Investigate related notions such as connections, parallel transport, curvature, and characteristic classes.

