Commutative nonassociative algebras, representations of finite groups and minimal cones

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Introduction

The most straightforward approach to an algebra (like $\mathbb{C}$ or $\mathbb{H}$) is to describe it by a multiplication table. Sometimes one define an algebra via an identity (like Lie and Jordan algebras). But, if you go beyond small dimensions, this approach is not very satisfactory.

Let $A$ be an commutative nonassociative algebra over $F$ decomposed into a direct sum

$$A = A_1 \oplus A_2 \oplus \ldots A_m.$$ 

The most interesting case is when $A_i$ are invariant subspaces of the adjoint operator

$$\text{ad}_c : x \rightarrow cx, \quad c \text{ being an idempotent in } A.$$ 

A fusion law or fusion rule is a map $\star : (i,j) \rightarrow i \star j$ such that

$$A_i A_j \subseteq \bigoplus_{k \in i \star j} A_k.$$ 

A fusion law is ‘nice’ if both $m$ and the cardinality of $i \star j$ are reasonably small. There are only few known examples with nice (i.e. graded) fusion laws. All of them have important features and connections to different areas of mathematics.
A key example: Jordan algebra of symmetric matrices

A commutative algebra $A$ is **Jordan** if $x^2(xy) = x(x^2y)$ for all $x, y \in A$.

Let $A$ be the 3-dimensional Jordan algebra of real symmetric $2 \times 2$-matrices with multiplication

$$x \circ y = \frac{1}{2}(xy + yx)$$

Note that $x \circ x = xx = x^2$. Set $c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

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$\Rightarrow \{e_1, e_2, e_3\}$ is an eigenbasis of $\text{ad}_c$ and

$$A = A_1 \oplus A_0 \oplus A\frac{1}{2}$$

with $A_1 = \langle e_1 \rangle$, $A_0 = \langle e_2 \rangle$ and $A\frac{1}{2} = \langle e_3 \rangle$.

This example illustrates an important general fact: $A$ satisfies the **Jordan-type** fusion law:

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Finite simple groups

Just as prime numbers, finite simple groups are the building blocks of finite groups.

**The Classification** (1981): any finite simple group occurs either in one of 18 regular infinite series like cyclic or Lie type groups, or it is one of the 26 sporadic groups.

- **Example.** The linear group $L_2(7) \cong L_3(2)$ is the second smallest non-abelian simple group (of order 168). It is the group of automorphisms of the Fano plane.

$L_2(7)$ is a $2A$-generated subgroup of the Monster, the largest sporadic group.

![Diagram of The Fano plane and Incidence graph](image-url)
The Monster Group $\mathbb{M}$ is the largest of the sporadic simple groups: its order is

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \cdot \approx 8 \cdot 10^{53}$$

196883 is the dimension of the first nontrivial representation of the Monster.

In 1978 McKay noticed that the Fourier coefficients of the modular function

$$j(\tau) = \frac{(\theta_2^3(\tau) + \theta_3^3(\tau) + \theta_4^3(\tau))^3}{8\eta(\tau)^24} = \sum_{n=-1}^{\infty} c_n q^n$$

$$= q^{-1} + 744 + (196883 + 1)q + \ldots, \quad q = e^{2\pi i \tau}$$

are linear combinations of the character degrees of the Monster! McKay and Thompson suggested that a natural graded module for $\mathbb{M}$ might exist.

Conway and Norton formulate 'Monstrous moonshine' (1979).

In 1988, Frenkel, Lepowsky & Meurman explicitly constructed a graded vertex operator algebra representation

$$V^\# = \bigoplus_{n \geq 0} V_n \quad \text{of } \mathbb{M} \text{ with } V_2 = V_\mathbb{M}.$$ 

In 1992 Borcherds finally settled the Moonshine conjecture (Fields medal 1998)
Griess algebra

- The Monster was constructed in 1982 by Griess as a group of automorphisms of a **non-associative commutative unital algebra** $V_M$ of dimension $1 + 196883$ over $\mathbb{R}$, now called the Griess algebra.

- The Griess algebra admits a natural invariant associating bilinear form, i.e.

  $$\langle xy, z \rangle = \langle x, yz \rangle, \quad \forall x, y, z \in V_M$$

- The Monster is generated by its involutions (the so-called (2A)-invoultions).

- There is a natural bijection between the (2A)-invoultions and a subset of idempotents (**axes**) in $V_M$.

- Any axis induces an orthogonal decomposition $A = A_1 \oplus A_0 \oplus A_{\frac{1}{4}} \oplus A_{\frac{1}{32}}$ with the **Ising fusion laws**:

  $\begin{array}{c|c|c|c|c}
  * & 1 & 0 & \frac{1}{4} & \frac{1}{32} \\
  \hline
  1 & 1 & 0 & \frac{1}{4} & \frac{1}{32} \\
  0 & 0 & 0 & \frac{1}{4} & \frac{1}{32} \\
  - & - & - & - & - \\
  \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1, 0 & \frac{1}{32} \\
  \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & 1, 0, \frac{1}{4} \\
  \end{array}$

  $\begin{array}{c|c|c|c|c|c|c|c}
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  + & + & + & + & + & - \\
  + & + & + & + & + & - \\
  + & + & + & + & + & - \\
  - & - & - & - & - & + \\
  \end{array}$
Griess algebra as a Majorana algebra

\[ A_+ := A_1 \oplus A_0 \oplus A_{\frac{1}{4}} \] is a subalgebra

There is a natural grading \( A = A_+ \oplus A_- \) with \( A_- := A_{\frac{1}{32}} \)

A key element of the construction: the Majorana involution

\[ x \rightarrow x^\tau := x_1 + x_0 + x_{\frac{1}{4}} - x_{\frac{1}{32}} \]

Furthermore, if \( x_{\frac{1}{32}} = 0 \) then \( x \rightarrow x^\sigma := x_1 + x_0 - x_{\frac{1}{4}} \) is also an algebra involution.

The above algebra structure determines the Monster as an isomorphism group.
In his talks on G-NA-VOA project, Robert Griess pointed out that a good theory of commutative nonassociative algebras generalizing the Monster algebra is highly relevant to developing a unified framework of finite simple groups, especially the sporadic groups.

A starting point is the commutative nonassociative algebra $V_2 = V_M$ in the VOA $V^\#$.

Via foundational work of M. Miyamoto (1995) and later by A.A. Ivanov (2007), the Griess algebra $V_2$ has been understood as a commutative non-associative algebra generated by a finite set of *primitive* idempotents (i.e. the dimension of the 1-eigenspace is one) satisfying certain axioms, the so-called *Majorana algebras*.

An algebra is *Majorana* if it possesses a positive definite associating bilinear form $\langle x, y \rangle$ and it is generated (as an algebra) by a finite set of idempotents with the $(1, 0, \frac{1}{4}, \frac{1}{32})$-Peirce decomposition and the Ising fusion laws. Some further (somewhat subtle) axioms like the Norton inequality are also required.

(Ivanov, *The Monster Group and Majorana Involutions*, CTM, 2009) proves that the Griess algebra of the Monster is a Majorana algebra and that the action of the Monster on its algebra realizes a Majorana representation of the Monster.
Axial algebras

- A further, radical, step was taken by J. Hall, S. Shpectorov & F. Rehren (2015), they put the previous work into a natural context of axial algebras.

- An axial algebra over the field $\mathbb{F}$ is a commutative algebra generated by idempotents whose adjoint action has the same multiplicity-free minimal polynomial.

Axial algebras include Jordan algebras, the Matsuo algebras for groups of 3-transpositions, as well as the 196884-dimensional Griess algebra.

- Recall that $V_M$ is generated by $(2A)$-idempotents satisfying the Ising fusion laws. J. Hall, S. Shpectorov and F. Rehren, (2015) completely characterized axial algebras satisfying the Jordan type fusion laws for $\eta \neq \frac{1}{2}$

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The case $\eta = \frac{1}{2}$ is singular (contains also Jordan algebras) and is not settled yet.
How incident is a particular Peirce spectrum?

We have seen that

- for Jordan algebras $\text{spec}(c) \subset \{1, 0, \frac{1}{2}\}$
- while for the Griess algebra $\text{spec}(c) \subset \{1, 0, \frac{1}{4}, \frac{1}{32}\}$

(Krasnov Ya., V.T., 2018) proved that a generic algebra does not contain $\frac{1}{2}$ in its spectrum. On the other hand, we have

**Theorem (V.T., 2018, submitted)**

Let $A$ be a finite dimensional commutative nonassociative algebra over a field of characteristic $\neq 2, 3$ and let $A$ satisfy a nontrivial identity $P(z) = 0$. Then

(A) $\frac{1}{2} \in \text{spec}(c)$ for any idempotent $c \neq 0$;

(B) if $c$ is semi-simple and $\lambda$ is single root of the Peirce polynomial $\varrho(P, t)$ then

$$A_\lambda A_{\frac{1}{2}} \subset A_{\frac{1}{2}} := \bigoplus_{\substack{\nu \in \text{spec}(c) \\ \nu \neq \lambda}} A_{\nu}.$$ 

In particular, $A_{\frac{1}{2}} A_{\frac{1}{2}} \subset A_{\frac{1}{2}}$. 
A toy example: the Harada algebra

Motivated by Griess’ construction of the Monster, the same year Koichiro Harada defined a 6-dimensional algebra $H$ whose automorphism group is $L_2(7)$ as follows:

$c_1, \ldots, c_6$ generate $H$ as a vector space with multiplication

$$c_i c_j = \begin{cases} 
  c_i & \text{if } i = j \\
  c_i \wedge j, & \text{if } i \neq j,
\end{cases}$$

and an additional identity

$$c_0 + c_1 + \ldots + c_6 = 0.$$

For example, $c_1 c_2 = c_1 \wedge 2 = c_3$.

- Each line is a subalgebra!
- Any automorphism $g \in \text{Aut}(H)$ preserves idempotents: $x^2 = x \Rightarrow g(x)^2 = g(x)$.
- Hence, $\text{Aut}(H)$ acts on $\Phi$ by permutations.
- Since $\{c_i\}$ generates $H$, the group $\text{Aut}(H)$ acts faithfully on $\Phi$.
- Since each line is a subalgebra, $\text{Aut}(H)$ act naturally on lines.

This proves that

$$\text{Aut}(H) = \text{Aut}(\Phi) = L_2(7).$$
The Harada algebra

- $\Phi$ has totally
  \[\#\text{Idem}(H) = 2^{\dim H} - 1 = 63 = 7 + 7 + 21 + 28\] nonzero idempotents.

- There are exactly five orbits of the $\text{Aut}(H)$-action on $\text{Idem}(H)$:
  - 7 points: $c_i$
  - 7 lines: $\frac{1}{3}(c_i \land j + c_i + c_j)$,
  - 21 flags: $\frac{1}{3}(2c_i \land j - c_i - c_j)$;
  - 28 anti-flags: $-3(c_i \land j + c_i + c_j) - 5c_m$.

**Theorem** (Ya. Krasnov, T., 2018, in preparation) Idempotents in each orbit have the same spectrum and the only graded (Ising type) fusion laws occur for flags:

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<td>1, 0, $\frac{1}{4}$</td>
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A **minimal surface** minimizes locally the area functional. Geometrically, this means that the **mean curvature** $= 0$. Analytically, a solution of certain nonlinear PDE.

A **minimal cone** is a typical singularity of a minimal surface. The most of known minimal cones are algebraic, i.e. zero level sets of a homogeneous polynomial $u \in \mathbb{R}[x_1, \ldots, x_n]$. Degree 2 minimal cones are well-known. An example is the Clifford-Simon cone given by

$$u(x) := (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

which is also known as the norm for split octonions. This cone played a crucial role in the solution of the famous Bernstein problem and counterexamples in higher dimensions.
We shall concern with \textit{cubic} minimal cones, i.e. cones of the smallest nontrivial degree 3.

**Hsiang’s problem (1967).** How to characterize cubic minimal cones? Can one at least characterize all cubic polynomial solutions of

\[
|Du|^2 \Delta u - \frac{1}{2} \langle Du, D|Du|^2 \rangle = \lambda |x|^2 u,
\]

or, more explicitly,

\[
(u_{x_1}^2 + \ldots + u_{x_n}^2)(u_{x_1}x_1 + \ldots + u_{x_n}x_n) - \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i} x_j = \lambda (x_1^2 + \ldots + x_n^2) \cdot u(x)
\]
Some explicit examples

- \( u = \text{Re}(z_1 z_2) z_3 \), the triality polynomials in \( \mathbb{R}^{3d} \), where \( z_i \in \mathbb{R}, \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \)

- \( u(x) = \begin{vmatrix}
\frac{1}{\sqrt{3}} x_1 + x_2 & x_3 & x_4 \\
x_2 & \frac{-2}{\sqrt{3}} x_1 & x_5 \\
x_4 & x_5 & \frac{1}{\sqrt{3}} x_1 - x_2 \\
\end{vmatrix} \)

  the generic norm in the Jordan algebra of \( 3 \times 3 \) symmetric matrices over \( \mathbb{R} \)

- \( u(x) = \begin{vmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6 \\
x_7 & x_8 & x_9 \\
\end{vmatrix} \)

  the generic norm in the Jordan algebra of \( 4 \times 4 \) symmetric \textit{traceless} matrices over \( \mathbb{R} \)

Thus, Hsiang cubics are nicely encoded by certain algebraic structures. Which ones?
A metrized algebra approach

Let $u$ be a cubic form on a vector space $V$ over $\mathbb{F}$ with an inner product $\langle x, y \rangle$. Linearize $u$ to get a symmetric trilinear form $\tilde{u}(x, y, z)$ such that

$$\tilde{u}(x, x, x) = 6u(x)$$

Define a multiplication $(x, y) \rightarrow xy$ on $V$ by the duality:

$$\tilde{u}(x, y, z) = \langle xy, z \rangle, \quad \forall z \in V.$$ 

Thus obtained algebra $V(u)$ is commutative and metrized:

$$\langle xy, z \rangle = \langle x, yz \rangle$$

and the cubic form is recovered by $u(x) = \frac{1}{6} \langle x, x^2 \rangle$. We also have

- $x^2 = 2 \text{grad } u(x)$
- $x$ is an idempotent in $V$ iff $kx$ is a stationary point of $u$ ($x \parallel \text{grad } u(x)$).
- $xy = \text{Hess } u(x)y \Rightarrow \text{ad}_x = \text{Hess } u(x)$
- $\text{ad}_x$ is self-adjoint: $\langle \text{ad}_x y, z \rangle = \langle y, \text{ad}_x z \rangle$
- Any idempotent is semisimple!
Algebras of minimal cones

In the above setup,

\[ \langle Du, Du \rangle \Delta u - \langle Du, \text{Hess}(u)Du \rangle = \lambda |x|^2 u \]

yields

\[ \langle \frac{x^2}{2}, \frac{x^2}{2} \rangle \text{tr}(\text{ad}_x) - \frac{1}{2} \langle \frac{x^2}{2}, \frac{x^3}{2} \rangle = \frac{\lambda}{6} \langle x, x \rangle \langle x, x^2 \rangle. \]

One can show that \( u \) is harmonic (non-trivial!), i.e.

\[ \text{ad}_x \quad \text{is trace less!} \]

This implies that to any cubic cone one can attach a commutative metrized algebra with

\[
\begin{cases}
\langle x^2, x^3 \rangle = \langle x, x \rangle \langle x^2, x \rangle \\
\text{tr}(\text{ad}_x) = 0.
\end{cases}
\]

(1)

Conversely: if \( A \) satisfies (1) then \( u(x) = \frac{1}{6} \langle x, x^2 \rangle \) generates a minimal cone.

We arrive at classification of all algebras satisfying (1). We call these \textbf{Hsiang algebras}. 
Two basic examples in dimensions 2 and 3

**Example 1.** Let $A$ be the algebra on $\mathbb{R}^2$ generated by idempotents $c_0, c_1, c_2$ with

\[
c_0 + c_1 + c_2 = 0
\]

Then for any triple $\{i, j, k\} = \{1, 2, 3\}$

\[
c_k = c_k^2 = (-c_i - c_j)^2 = c_i + c_j + 2c_i c_j = -c_k + 2c_i c_j
\]

\[
\Rightarrow c_i c_j = c_k \quad \Rightarrow \quad c_k (c_i - c_j) = -1(c_i - c_j).
\]

This implies that $A$ is a Hsiang algebra, $A = A_1 \oplus A_{-1}$ with $\mathbb{Z}_2$-graded fusion laws

\[
\begin{array}{c|cc}
\ast & 1 & -1 \\
1 & 1 & -1 \\
-1 & -1 & 1 \\
\end{array}
\]

The **minimal cone** is given by $x_1^2 x_2 = 0$, i.e. pair of two orthogonal lines in $\mathbb{R}^2$
Two basic examples in dimensions 2 and 3

Example 2. Let $A$ be generated by idempotents $c_0, c_1, c_2, c_3$ in $\mathbb{R}^3$ subject to

$$(c_i + c_j)^2 = 0 \quad \text{for } i \neq j$$

Then similarly to the above, one easily verifies that

$$A = A_1 \oplus A_{-\frac{1}{2}}, \quad \dim A_1 = 1, \quad \dim A_{-\frac{1}{2}} = 2.$$ 

Then $A$ is a Hsiang algebra with fusion laws

$$
\begin{array}{c|cc}
\ast & 1 & -\frac{1}{2} \\
1 & 1 & -\frac{1}{2} \\
-\frac{1}{2} & -\frac{1}{2} & 1, -\frac{1}{2}
\end{array}
$$

After a rank one perturbation $A$ becomes a Jordan algebra of Clifford type. The minimal cone is given by $x_1x_2x_3 = 0$, i.e. the triple of coordinate planes in $\mathbb{R}^3$. 
**Definition.** A commutative algebra $V$ with associating form $\langle x, y \rangle$ is called **polar** if
- there is a $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$,
- $V_0V_0 = 0$,
- $x(xy) = |x|^2y$ for $x \in V_0$ and $y \in V_1$.

Any polar algebra is Hsiang!

**Remark.** Can be simply constructed using **symmetric Clifford systems**, i.e. symmetric matrices

$$A_i^2 = I, \quad A_iA_j + A_jA_i = 0, \quad i \neq j$$

This yields an obstruction:

$$q \leq 1 + \rho(p),$$

where $\rho(m) = 8a + 2^b$, if $m = 2^{4a+b} \cdot \text{odd}$, $0 \leq b \leq 3$ is the **Hurwitz-Radon function**.

**Definition.** A Hsiang algebra $V$ isomorphic to a polar algebra is of **Clifford type**; otherwise it is **exceptional**.
Any commutative **pseudocomposition algebra**, i.e. an algebra with

\[ x^3 = |x|^2 x, \]
\[ \text{tr} \text{ad}_x = 0 \]

is an **exceptional** Hsiang algebra.

We prove that there are only finitely many exceptional Hsiang algebras!
Many classification results reminiscence the famous Cartan-Killing classification: there are several ‘nice’ series and finitely many exceptional (sporadic) elements.

- the $A, B, C, D$-series of simple Lie groups and the exceptional Lie groups $G_2, F_4, E_6, E_7,$ and $E_8$;

- special Jordan algebras (obtained from associative algebras) vs the Albert 27-dimensional exceptional Jordan algebra

- the classification of finite simple groups

The fundamental question is why exceptional (sporadic) objects do really exist?
Finiteness of exceptional Hsiang algebras

The Peirce decomposition

Let $c$ be an idempotent. Then

$$A = A_1 \oplus A_{-1} \oplus A_{-\frac{1}{2}} \oplus A_{\frac{1}{2}}$$

where $A_1 = \mathbb{R}c$ (i.e. $c$ is primitive!) satisfying the Hsiang algebras fusion laws:

\[
\begin{array}{c|cccc}
\ast & 1 & -1 & -\frac{1}{2} & \frac{1}{2} \\
1 & 1 & -1 & -\frac{1}{2} & \frac{1}{2} \\
-1 & -1 & 1 & \frac{1}{2} & -\frac{1}{2}, \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1, -\frac{1}{2} & -1, \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}, \frac{1}{2} & -1, \frac{1}{2} & 1, -1, -\frac{1}{2}
\end{array}
\]

Two distinguished subalgebras: $A_1 \oplus A_{-1}$ (carries a hidden Clifford algebra structure) and $A_1 \oplus A_{-\frac{1}{2}}$ (carries a hidden rank 3 Jordan algebra structure)

Primitive idempotents $w$ in the hidden Jordan algebra ($w \ast w = w$) are exactly 2-nilpotents in $V$ ($w^2 = w$) with the fusion rules
Finiteness of exceptional Hsiang algebras

Main Theorem (V.T., 2014, 2016)
(i) Given an idempotent \( c \in A \), the new algebra on \( \Lambda_c = (A_1 \oplus A_{-1/2}, \bullet) \) with

\[
x \bullet y = \frac{1}{2} xy + \langle x, c \rangle y + \langle y, c \rangle x - 2 \langle xy, c \rangle c.
\]

is a Euclidean Jordan algebra of rank 3 with unit \( c^* = 2c \).

(ii) \( A \) is exceptional if and only if the Jordan algebra \( \Lambda_c \) is simple.

(iii) There holds

\[
\dim A_{-1} - 1 \leq \rho(\dim A_{-1} + \dim A_{-1/2} - 1),
\]

where \( \rho \) is the Hurwitz-Radon function.

(iv) There are at most 16 possible classes of exceptional Hsiang algebras:

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<td>5</td>
<td>9</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \dim A_{-1/2} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>5</td>
<td>8</td>
<td>8</td>
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<td>8</td>
<td>8</td>
<td>8</td>
<td>14</td>
<td>14</td>
<td>26</td>
</tr>
</tbody>
</table>

unsettled
Theorem (Existence)
Denote by $V(u)$ the algebra on a vector inner product space $V$ generated by the cubic form $u$. Then $A = V(u)$ where

- $\dim A_{-\frac{1}{2}} = 0$, $\dim A_{-1} = d + 1$, and $u = \frac{1}{6}\langle z, z^2 \rangle$, $V = \mathcal{H}_3(\mathbb{K}_d) \oplus \mathbb{R}e$, $d = 0, 1, 2, 4, 8$.

- $\dim A_{-1} = 0$, $\dim A_{-\frac{1}{2}} = \frac{3}{2}d + 2$, and $u = \frac{1}{12}\langle z^2, 3\bar{z} - z \rangle$, where $z \mapsto \bar{z}$ is the natural involution on $V = \mathcal{H}_3(\mathbb{K}_d)$, $d = 2, 4, 8$.

- $\dim A_{-1} = 1$, $\dim A_{-\frac{1}{2}} = 3d + 2$, and $u(z) = \text{Re}\langle z, z^2 \rangle$, where $z \in V = \mathcal{H}_3(\mathbb{K}_d) \otimes \mathbb{C}$, $d = 1, 2, 4, 8$.

- $\dim A_{-1} = 4$ and $\dim A_{-\frac{1}{2}} = 5$, where $u = \frac{1}{6}\langle z, z^2 \rangle$ on $\mathcal{H}_3(\mathbb{K}_8) \oplus \mathcal{H}_3(\mathbb{K}_1)$.
Epilogue: Nonassociative algebras and singular solutions

- Evans, Crandall, Lions, Jensen, Ishii: If $\Omega \subset \mathbb{R}^n$ is bounded with $C^1$-boundary, $\phi$ continuous on $\partial \Omega$, $F$ uniformly elliptic operator then the Dirichlet problem

$$F(D^2 u) = 0, \quad \text{in} \quad \Omega$$
$$u = \phi \quad \text{on} \quad \partial \Omega$$

has a unique viscosity solution $u \in C(\Omega)$;

- Krylov, Safonov, Trudinger, Caffarelli, early 80's: the solution is always $C^{1,\varepsilon}$

- Nirenberg, 50's: if $n = 2$ then $u$ is classical ($C^2$) solution

- Nadirashvili, Vlăduț, 2007-2011: if $n \geq 12$ then there are solutions which are not $C^2$.

**Theorem (Nadirashvili-V.T.-Vlăduț, 2012, 2015)** The function $w(x) := \frac{u_1(x)}{|x|}$ where

$$u_1(x) = x_5^3 + \frac{3}{2} x_5(x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2) + \frac{3\sqrt{3}}{2} x_4(x_2^2 - x_1^2) + 3\sqrt{3} x_1 x_2 x_3,$$

is a singular viscosity solution of the uniformly elliptic Hessian equation

$$(\Delta w)^5 + 2^{12} 3^2 (\Delta w)^3 + 2^{12} 3^5 \Delta w + 2^{15} \det D^2(w) = 0,$$

This also gives the best possible dimension $(n = 5)$ where homogeneous order 2 real analytic functions in $\mathbb{R}^n \setminus \{0\}$. 

Vladimir G. Tkachev

Linköping, September 27th, 2018 (26 of 29)
Some important questions remain unanswered

How incident (important, relevant) that the certain *commutative non-associative* algebraic structures coming from finite simple groups, geometry of minimal cones and PDEs (truly viscosity solutions)

- have a distinguished Peirce spectrum
- have distinguished (in particular, graded) fusion rules
- are axial
- are metrized (i.e. carrying an associating symmetric bilinear form)?


THANK YOU FOR YOUR ATTENTION!