On classification of $(n+1)$-dimensional n-Hom-Lie algebras

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Hom-Lie algebra appeared first in the process of studying the general quasi-deformations and discretizations of Lie algebras of vector fields using twisted derivations (Hartwig, Larsson, Silvestrov)

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- Hartwig J. T., Larsson D., Silvestrov S. D., Deformations of Lie algebras using σ-derivations (2003)
- Larsson D., Silvestrov S., Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities (2005)
- Makhlouf A., Silvestrov S. D., Hom-algebra structures (2006)

The concept of n-Lie algebras, also known as Nambu-Lie algebras is a generalization of Lie algebras to the n-ary case, the Jacobi identity is considered here as the fact that the adjoint maps are derivations. They were defined and studied both as purely algebraic structures (Filippov, Kasymov, ...) and related to Nambu mechanics (Takhtajan,...). They found afterwards several other applications in physics.

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- Nambu, Y.: Generalized Hamiltonian dynamics (1973)
- Takhtajan, L. A.: On foundation of the generalized Nambu mechanics (1994)
- Filippov V. T., n-Lie algebras (1985)
- Kasymov Sh. M., Theory of n-Lie algebras (1987)

Later on, n-ary generalizations of Hom-algebra, both of Lie and associative type were introduced (Ataguema, Makhlouf, Silvestrov) and several aspects of their structures have been and are still studied (cohomology, deformations, representations, extensions, ...)

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- Ataguema H., Makhlouf A., Silvestrov S., Generalization of n-ary Nambu algebras and beyond (2009)
- Yau D., On n-ary Hom-Nambu and Hom-Nambu-Lie algebras (2012)

Definition

An n-Hom-Lie algebra $(A, [\cdot, \ldots, \cdot], (\alpha_i)_{1\leq i\leq n-1})$ is a linear space equipped with an n-ary skew-symmetric operation and $(n-1)$ linear maps satisfying, for all $x_1, \ldots, x_{n-1}, y_1, \ldots, y_n \in A$,

$$[\alpha_1(x_1), \ldots, \alpha_{n-1}(x_{n-1}), [y_1, \ldots, y_n]] = \sum_{i=1}^{n} [\alpha_1(y_1), \ldots, \alpha_{i-1}(y_{i-1}), [x_1, \ldots, x_{n-1}, y_i], \alpha_i(y_{i+1}), \ldots, \alpha_{n-1}(y_n)]. \quad (1)$$

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In the case where $\alpha_1 = \alpha_2 = ... = \alpha_{n-1} = \alpha$, we shall use the notation $(A, [\cdot, \dots, \cdot], \alpha)$ instead. Our study is mostly about this case.

Definition

Let $(A, [\cdot, \ldots, \cdot]_A, (\alpha_i)_{1 \leq i \leq n-1})$ and $(B, [\cdot, \ldots, \cdot]_B, (\beta_i)_{1 \leq i \leq n-1})$ be two n-Hom-Lie algebras. An n-Hom-Lie algebra morphism is a linear map $f : A \to B$ satisfying:

$$f\left([x_1, ..., x_n]_A\right) = [f(x_1), ..., f(x_n)]_B$$

and

$$f \circ \alpha_i = \beta_i \circ f.$$

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A linear map $f : A \to B$ satisfying only the first condition is said to be a weak morphism.

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An n-Hom-Lie algebra $(A, [\cdot, \ldots, \cdot], \alpha)$ is said to be multiplicative if α is an n-Hom-Lie algebra morphism, if moreover α is invertible, $(A, [\cdot, \ldots, \cdot], \alpha)$ is said to be regular

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Definition

Let $(A, [\cdot, \dots, \cdot], \{\alpha_i\}_{1\leq i\leq n-1})$ be an n-Hom-Lie algebra and let S be a linear subspace of A. B is said to be an ideal (or Hom-ideal) if it is invariant under all $\alpha_i, 1 \leq i \leq n-1$ and for all $s \in S, x_1, ..., x_{n-1} \in A$, $[x_1, ..., x_{n-1}, s] \in S$.

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If this condition is satisfied but S is not invariant under $\alpha_i, 1 \leq i \leq n-1$, it is said to be a weak ideal.

Definition

Let $(A, [\cdot, \dots, \cdot], (\alpha_i)_{1\leq i\leq n-1})$ be an n-Hom-Lie algebra, the k-derived series of the ideal I is defined by

$$D_k^0(I) = I \text{ and } D_k^{p+1} = \Big[\underbrace{D_k^p(I), \dots, D_k^p(I)}_{k}, A, \dots, A\Big],$$

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The ideal I is said to be k-solvable (resp. k-nilpotent) if there exists $r \in \mathbb{N}$ such that $D_k^r(I) = \{0\}$ (resp. $C_k^r(I) = \{0\}$).

A linear map α is fully determined by its matrix on the considered basis, and a skew-symmetric n-ary multi-linear bracket is fully determined by $[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}]$ for all $1 \leq i \leq n+1$

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$$\begin{aligned}
[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}] &= (-1)^{n+1+i} w_i, \\
w_i &= \sum_{p=1}^{n+1} b_{p,i} e_p \\
(w_1, \ldots, w_{n+1}) &= (e_1, \ldots, e_{n+1}) B, \qquad (2)
\end{aligned}$$

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for $B = (b_{i,j})_{1 \leq i,j \leq n+1}$.

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for $B = (b_{i,j})_{1 \leq i,j \leq n+1}$.
In term of structure constants, one can write :

$$w_i = \sum_{p=1}^{n+1} (-1)^{n+1+i} c(1, ..., i-1, i+1, ..., n+1, p) e_p$$

In all the following, $(A, [\cdot, \ldots, \cdot], \alpha)$ is an n-ary skew-symmetric algebra of dimension $n+1$ with a linear map α and $(e_i)_{1 \leq i \leq n+1}$ is a basis of A.

Proposition

Let $\mathcal{A}_1 = (A, [\cdot, \ldots, \cdot]_1, \alpha_1)$ and $\mathcal{A}_2 = (A, [\cdot, \ldots, \cdot]_2, \alpha_2)$ be two $(n+1)$-dimensional n-ary skew-symmetric Hom-algebras represented by matrices $[\alpha_1]$, B_1 and $[\alpha_2]$, B_2 respectively. The Hom-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic if and only if there exists an invertible matrix T satisfying the following conditions:

$$B_2 = \det(T)^{-1} T B_1 T^T,$$

$$[\alpha_2] = T[\alpha_1]T^{-1}.$$

We give now systems of equations equivalent to the Hom-Nambu-Filippov identity for different cases of α, together with its solution in some of the cases, We give now systems of equations equivalent to the Hom-Nambu-Filippov identity for different cases of α, together with its solution in some of the cases,first is when α is diagonalizable

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Proposition

If α is invertible, then $[\cdot,\ldots,\cdot]$ satisfies the Hom-Nambu-Filippov identity if and only if, $\forall 1 \leq i < j < k \leq n+1$

$$(\lambda_i b_{j,i} - \lambda_j b_{i,j}) w_k + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) w_j + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) w_i = 0, \quad (3)$$

which is also equivalent to the following system, obtained by using the coordinates in the basis $(e_i)_{1\leq i\leq n+1}$:
$\forall 1 \leq i,j,k,p \leq n+1; i < j < k,$

$$(\lambda_i b_{j,i} - \lambda_j b_{i,j}) b_{p,k} + (\lambda_k b_{i,k} - \lambda_i b_{k,i}) b_{p,j} + (\lambda_j b_{k,j} - \lambda_k b_{j,k}) b_{p,i} = 0.$$

Proposition

If $Rank(B) \geq 3$, then the Hom-Nambu-Filippov identity holds if and only if $\lambda_i b_{j,i} - \lambda_j b_{i,j} = 0, \forall i, j$, that is $B[\alpha]^T = [\alpha]B^T$ or equivalently that $B[\alpha]^T$ is symmetric, or in other words, the matrix B takes the form

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$$\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & \dots & b_{1,n+1} \\ \frac{\lambda_2 b_{1,2}}{\lambda_1} & b_{2,2} & b_{2,3} & \dots & \dots & b_{2,n+1} \\ \frac{\lambda_3 b_{1,3}}{\lambda_1} & \frac{\lambda_3 b_{2,3}}{\lambda_2} & b_{3,3} & \dots & \dots & b_{3,n+1} \\ \vdots & \vdots & \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \ddots & \vdots \\ \frac{\lambda_{n+1} b_{1,n+1}}{\lambda_1} & \frac{\lambda_{n+1} b_{2,n+1}}{\lambda_2} & \frac{\lambda_{n+1} b_{3,n+1}}{\lambda_3} & \dots & \frac{\lambda_{n+1} b_{n,n+1}}{\lambda_n} & b_{n+1,n+1} \end{pmatrix},$$

in any basis where α is diagonal, where $\lambda_i, 1 \leq i \leq n+1$ are the eigenvalues of α.

If $\dim\ker\alpha = 1$, let $\lambda_1 = 0$.

$$\lambda_k b_{1,k} w_j - \lambda_k b_{j,k} w_1 - \lambda_j b_{1,j} w_k + \lambda_j b_{k,j} w_1 = 0, \quad \forall 1 < j < k \leq n+1.$$

If $\dim \ker \alpha = 2$, let $\lambda_1 = \lambda_2 = 0$.

$$b_{1,k} w_2 - b_{2,k} w_1 = 0, \text{ for all } 3 \leq k \leq n+1.$$

If $\dim\ker\alpha = 2$, let $\lambda_1 = \lambda_2 = 0$.

$$b_{1,k}w_2 - b_{2,k}w_1 = 0, \text{ for all } 3 \leq k \leq n+1.$$

Note that in this case, the non-zero eigenvalues do not appear in the equations, which means that if a given skew-symmetric bracket satisfies the Hom-Nambu-Filippov identity for a diagonalizable α with kernel of dimension 2, it would satisfy it for any such a linear map.

For $n = 3$, α is represented in this case by the matrix
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$
, with $\lambda_3, \lambda_4 \neq 0$ and we get the following list of brackets (represented by matrices)

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$$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & 0 & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & 0 & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$$

$$\begin{pmatrix} \frac{c(1,2,3,1)c(1,3,4,1)}{c(1,2,3,2)} & c(1,3,4,1) & -\frac{c(1,2,3,1)c(1,2,4,2)}{c(1,2,3,2)} & c(1,2,3,1) \\ \frac{c(1,2,3,1)c(1,3,4,2)}{c(1,2,3,2)} & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ \frac{c(1,2,3,1)c(1,3,4,3)}{c(1,2,3,2)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ \frac{c(1,2,3,1)c(1,3,4,4)}{c(1,2,3,2)} & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$$

$\begin{pmatrix} \frac{c(1,2,4,1)c(1,3,4,1)}{c(1,2,4,2)} & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ \frac{c(1,2,4,1)c(1,3,4,2)}{c(1,2,4,2)} & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ \frac{c(1,2,4,1)c(1,3,4,3)}{c(1,2,4,2)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ \frac{c(1,2,4,1)c(1,3,4,4)}{c(1,2,4,2)} & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$\begin{pmatrix} -c(2,3,4,1) & 0 & -c(1,2,4,1) & c(1,2,3,1) \\ -c(2,3,4,2) & 0 & 0 & 0 \\ -c(2,3,4,3) & 0 & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & 0 & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
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$$(b_{k-1,i} - b_{i-1,k})b_{p,n+1} - b_{n+1,i}b_{p,k-1} + b_{n+1,k}b_{p,i-1} = 0,$$

for all $1 \leq i, k, p \leq n+1, i < k$.

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For $n = 3$, α is represented in this case by the matrix
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$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ -c(2,3,4,4) & 0 & 0 & 0 \end{pmatrix}$
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$\begin{pmatrix} s_1 & s_5 & -c(1,2,4,1) & c(1,2,3,1) \\ s_2 & s_6 & s_{10} & c(1,2,3,2) \\ s_3 & s_7 & \frac{c(1,2,4,4)^2}{c(1,2,3,4)} + c(1,2,3,2) - c(1,3,4,4) & c(1,2,3,3) \\ s_4 & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$\begin{pmatrix} t_1 & \frac{c(1,2,3,1)c(1,3,4,4)}{c(1,2,3,4)} & \frac{c(1,2,3,2)c(1,3,4,4)}{c(1,2,3,4)} - \frac{c(1,3,4,4)^2}{c(1,2,3,4)} & c(1,2,3,1) \\ t_2 & \frac{c(1,2,3,2)c(1,3,4,4)}{c(1,2,3,4)} & -c(1,2,4,2) & c(1,2,3,2) \\ t_3 & \frac{c(1,2,3,3)c(1,3,4,4)}{c(1,2,3,4)} & c(1,2,3,2) - c(1,3,4,4) & c(1,2,3,3) \\ t_4 & c(1,3,4,4) & 0 & c(1,2,3,4) \end{pmatrix}$

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$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & 0 & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ -c(2,3,4,4) & c(1,3,4,4) & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} -c(2,3,4,1) & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ -c(2,3,4,2) & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ -c(2,3,4,3) & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ -c(2,3,4,4) & c(1,3,4,4) & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} \frac{c(1,2,3,1)^2}{c(1,2,3,4)} & c(1,3,4,1) & -\frac{c(1,2,3,1)c(1,2,4,4)}{c(1,2,3,4)} & c(1,2,3,1) \\ \frac{c(1,2,3,1)c(1,2,3,2)}{c(1,2,3,4)} & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ \frac{c(1,2,3,1)c(1,2,3,3)}{c(1,2,3,4)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ c(1,2,3,1) & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$

$\begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ \frac{c(1,2,3,2)c(1,2,4,1)}{c(1,2,4,4)} & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ \frac{c(1,2,3,3)c(1,2,4,1)}{c(1,2,4,4)} & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & c(1,3,4,1) & 0 & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & c(1,2,3,2) \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & c(1,2,3,3) \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & c(1,2,3,4) \end{pmatrix}$
$\begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$

We consider now the case from the previous table where the bracket is given, in the basis (e_i) by the matrix

$$B = \begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$$

We consider now the case from the previous table where the bracket is given, in the basis (e_i) by the matrix

$$B = \begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix}$$ that is

$$[e_1, e_2, e_3] = 0$$
$$[e_1, e_2, e_4] = c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4$$
$$[e_1, e_3, e_4] = c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4$$
$$[e_2, e_3, e_4] = 0.$$

We consider now the case from the previous table where the bracket is given, in the basis (e_i) by the matrix

$$B = \begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix} \text{ that is}$$

$$[e_1,e_2,e_3] = 0$$
$$[e_1,e_2,e_4] = c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4$$
$$[e_1,e_3,e_4] = c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4$$
$$[e_2,e_3,e_4] = 0.$$

and the twisting map α is defined, in the same basis, by

$$[\alpha] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

A 3-Hom-Lie algebra of this class is multiplicative if and only if

$$c(1,2,4,3)=0; c(1,2,4,4)=0; c(1,3,4,3)=0; c(1,3,4,4)=0.$$

That is, its bracket is of the form

$$\begin{aligned}
[e_1,e_2,e_3] &= 0 \\
[e_1,e_2,e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 \\
[e_1,e_3,e_4] &= c(1,3,4,1)e_1 + c(1,3,4,2)e_2 \\
[e_2,e_3,e_4] &= 0
\end{aligned}$$

We will first split this class into five subclasses of non-isomorphic 3-Hom-Lie algebras. The split shall be done following the differences in the k-derived series, central descending series and center of the algebra, since these are preserved by isomorphisms.

We will first split this class into five subclasses of non-isomorphic 3-Hom-Lie algebras. The split shall be done following the differences in the k-derived series, central descending series and center of the algebra, since these are preserved by isomorphisms.
define $d(p,q) = c(1,2,4,p)c(1,3,4,q) - c(1,2,4,q)c(1,3,4,p)$ with $1 \leq p,q \leq 4$, that is, $d(p,q)$ are all the potentially non-zero 2×2 minors of the matrix B defining the bracket.

The five subclasses are given with properties of their k-derived series, central descending series and center:

The five subclasses are given with properties of their k-derived series, central descending series and center:
1) 3-solvable of class 2, non-2-solvable, non-nilpotent, with trivial center.

$$[e_1, e_2, e_3] = 0$$
$$[e_1, e_2, e_4] = c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4$$
$$[e_1, e_3, e_4] = c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4$$
$$[e_2, e_3, e_4] = 0,$$

with $d(1,4) \neq 0$, in that case we have
$$Rank \begin{pmatrix} d(2,4) & d(3,4) \\ d(1,2) & d(1,3) \end{pmatrix} = 2.$$

2) 3-solvable of class 2, 2-solvable of class 3, non-nilpotent, with trivial center.

$$\begin{aligned}
[e_1,e_2,e_3] &= 0\\
[e_1,e_2,e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4\\
[e_1,e_3,e_4] &= \lambda c(1,2,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + \lambda c(1,2,4,4)e_4\\
[e_2,e_3,e_4] &= 0,
\end{aligned}$$

with $(c(1,2,4,1),c(1,2,4,4)) \neq (0,0)$ or

$$\begin{aligned}
[e_1,e_2,e_3] &= 0\\
[e_1,e_2,e_4] &= c(1,2,4,2)e_2 + c(1,2,4,3)e_3\\
[e_1,e_3,e_4] &= c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4\\
[e_2,e_3,e_4] &= 0,
\end{aligned}$$

such that $Rank\begin{pmatrix} d(2,4) & d(3,4)\\ d(1,2) & d(1,3)\end{pmatrix} = 1$.

3) 3-solvable of class 2, 2-solvable of class 2, non-nilpotent, with trivial center.

$$\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= c(1,2,4,2)e_2 + c(1,2,4,3)e_3 \\
[e_1, e_3, e_4] &= c(1,3,4,2)e_2 + c(1,3,4,3)e_3 \\
[e_2, e_3, e_4] &= 0
\end{aligned}$$

with $d(2,3) \neq 0$.

4) 3-solvable of class 2, 2-solvable of class 2, non-nilpotent, with 1-dimensional center.

$$
\begin{aligned}
&[e_1,e_2,e_3]=0\\
&[e_1,e_2,e_4]=c(1,2,4,1)e_1+c(1,2,4,2)e_2+c(1,2,4,3)e_3+c(1,2,4,4)e_4\\
&[e_1,e_3,e_4]=\lambda c(1,2,4,1)e_1+\lambda c(1,2,4,2)e_2+\lambda c(1,2,4,3)e_3+\lambda c(1,2,4,4)e_4\\
&[e_2,e_3,e_4]=0
\end{aligned}
$$

with $[e_1,e_2,e_4]\neq 0$ (that is not all $c(1,2,4,1),c(1,2,4,2),c(1,2,4,3),c(1,2,4,4)$ are zero) or

$$
\begin{aligned}
&[e_1,e_2,e_3]=0\\
&[e_1,e_2,e_4]=0\\
&[e_1,e_3,e_4]=c(1,3,4,1)e_1+c(1,3,4,2)e_2+c(1,3,4,3)e_3+c(1,3,4,4)e_4\\
&[e_2,e_3,e_4]=0
\end{aligned}
$$

5) 3-solvable of class 2, 2-solvable of class 2, nilpotent of class 2, with 1-dimensional center.

$$\begin{aligned}
[e_1,e_2,e_3] &= 0\\
[e_1,e_2,e_4] &= c(1,2,4,2)e_2 + c(1,2,4,3)e_3\\
[e_1,e_3,e_4] &= \frac{-c(1,2,4,2)^2}{c(1,2,4,3)}e_2 - c(1,2,4,2)e_3\\
[e_2,e_3,e_4] &= 0, \quad \text{where } c(1,2,4,3)\neq 0,
\end{aligned}$$

or

$$\begin{aligned}
[e_1,e_2,e_3] &= 0\\
[e_1,e_2,e_4] &= c(1,2,4,2)e_2 + \frac{-c(1,2,4,2)^2}{c(1,3,4,2)}e_3\\
[e_1,e_3,e_4] &= c(1,3,4,2)e_2 - c(1,2,4,2)e_3\\
[e_2,e_3,e_4] &= 0, \quad \text{where } c(1,3,4,2)\neq 0.
\end{aligned}$$

In the subclasses presented, cases 1 and 3 cannot be multiplicative. All the multiplicative 3-Hom-Lie algebras in the considered class are contained in the remaining subclasses:

2m) 3-solvable of class 2, 2-solvable of class 3, non-nilpotent, with trivial center

$$\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 \\
[e_1, e_3, e_4] &= c(1,3,4,1)e_1 + c(1,3,4,2)e_2 \\
[e_2, e_3, e_4] &= 0,
\end{aligned}$$

with
$d(1,2) = c(1,2,4,1)c(1,3,4,2) - c(1,2,4,2)c(1,3,4,1) \neq 0$.

4m) 3-solvable of class 2, 2-solvable of class 2, non-nilpotent, with 1-dimensional center

$$
\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 \\
[e_1, e_3, e_4] &= \lambda c(1,2,4,1)e_1 + \lambda c(1,2,4,2)e_2 \\
[e_2, e_3, e_4] &= 0
\end{aligned}
$$

5m) 3-solvable of class 2, 2-solvable of class 2, nilpotent of class 2, with 1-dimensional center

$$[e_1, e_2, e_3] = 0$$
$$[e_1, e_2, e_4] = 0$$
$$[e_1, e_3, e_4] = c(1,3,4,2)e_2$$
$$[e_2, e_3, e_4] = 0,$$

where $c(1,3,4,2) \neq 0$.

The condition $[\alpha] = P[\alpha]P^{-1}$ is satisfied if and only if P is of the form:

$$P = \begin{pmatrix} p(1,1) & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & p(2,3) & p(2,4) \\ 0 & 0 & p(3,3) & p(2,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}.$$

The condition $[\alpha] = P[\alpha]P^{-1}$ is satisfied if and only if P is of the form:

$$P = \begin{pmatrix} p(1,1) & 0 & 0 & p(1,4) \\ p(2,1) & p(3,3) & p(2,3) & p(2,4) \\ 0 & 0 & p(3,3) & p(2,3) \\ 0 & 0 & 0 & p(3,3) \end{pmatrix}.$$

Less possibilities for isomorphisms means more isomorphism classes, in particular, one cannot simply proceed by picking a new basis without checking that the basis change matrix is in the given form.

$\dim D_3^1(\mathcal{A}) = 2$, non-2-solvable, non-nilpotent, with trivial center.
$c(1,2,4,4) \neq 0$.

$$\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= e_4 \\
[e_1, e_3, e_4] &= c'(1,3,4,1)e_1 + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4 \\
[e_2, e_3, e_4] &= 0,
\end{aligned}$$

Two such algebras, given by structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ respectively are isomorphic if and only if $c'(1,3,4,3) = c''(1,3,4,3)$ and $c'(1,3,4,4) = c''(1,3,4,4)$ and $\frac{c'(1,3,4,1)}{c''(1,3,4,1)}$ is a square in $\mathbb{K}$.

$c(1,2,4,4) = 0$, $c(1,2,4,3) \neq 0$ and $c(1,2,4,3) \neq c(1,3,4,4)$. In this case $c(1,2,4,1)$ and $c(1,3,4,4)$ are non-zero since $d(1,4) \neq 0$.

$$\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= c'(1,2,4,1)e_1 + e_3 \\
[e_1, e_3, e_4] &= c'(1,3,4,4)e_4 \\
[e_2, e_3, e_4] &= 0,
\end{aligned}$$

Two such algebras, given by structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ respectively are isomorphic if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$.

$c(1,2,4,4)=0$, $c(1,2,4,3)\neq 0$ and $c(1,2,4,3)=c(1,3,4,4)$. In this case also $c(1,2,4,1)$ and $c(1,3,4,4)$ are non-zero since $d(1,4)\neq 0$.

$$
\begin{aligned}
[e_1,e_2,e_3] &= 0\\
[e_1,e_2,e_4] &= c'(1,3,4,3)e_3+e_4\\
[e_1,e_3,e_4] &= c'(1,2,4,1)e_1+e_3\\
[e_2,e_3,e_4] &= 0,
\end{aligned}
$$

Two such algebras, given by structure constants $(c'(i,j,k,p))$ and $(c''(i,j,k,p))$ respectively are isomorphic if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$.

$c(1,2,4,4)=0$ and $c(1,2,4,3)=0$. Similarly, in this case $c(1,2,4,1)$ and $c(1,3,4,4)$ are non-zero since $d(1,4)\neq 0$.

$$\begin{aligned}[e_1,e_2,e_3]&=0\\ [e_1,e_2,e_4]&=c'(1,2,4,1)e_1\\ [e_1,e_3,e_4]&=e_4\\ [e_2,e_3,e_4]&=0,\end{aligned}$$

Two such brackets given by the structure constants $(c'(i,j,k,p))$, $(c''(i,j,k,p))$ are isomorphic if and only if $\frac{c'(1,2,4,1)}{c''(1,2,4,1)}$ is a square in $\mathbb{K}$. In particular, If $c(1,2,4,1)$ is a square in $\mathbb{K}$, we get the following bracket

$$\begin{aligned}[e_1,e_2,e_3]&=0\\ [e_1,e_2,e_4]&=e_1\\ [e_1,e_3,e_4]&=e_4\\ [e_2,e_3,e_4]&=0,\end{aligned}$$

Examples

We consider the two following examples,

1.
$$[e_1, e_2, e_4] = e_4$$
$$[e_1, e_3, e_4] = c'(1,3,4,1)e_1 + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4$$
$$c'(1,3,4,1) \neq 0,$$

2.
$$[e_1, e_2, e_4] = e_4$$
$$[e_1, e_3, e_4] = c'(1,3,4,2)e_2 + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4$$
$$(c'(1,2,4,1), c'(1,2,4,4)) \neq (0,0)$$

$$D_3^1(\mathcal{A}) = D_2^1(\mathcal{A}) = \langle \{ [e_1, e_2, e_4] \, , [e_1, e_3, e_4] \} \rangle.$$

For both cases, $D_3^1(\mathcal{A})$ is not invariant under α, therefore, it is a weak ideal of $\mathcal{A}$ but not a Hom-ideal. Also, for example 1, $D_3^2(\mathcal{A}) = \{0\}$ and $D_2^2(\mathcal{A}) = D_2^1(\mathcal{A})$.

For example 2

$$D_2^2(\mathcal{A}) = \langle \{c'(1,3,4,2)[e_1,e_2,e_4] + c'(1,3,4,3)[e_1,e_3,e_4]\}\rangle$$
$$= \langle \{v\}\rangle \neq \{0\}$$

where $v = c'(1,3,4,3)c'(1,3,4,2)e_2 + c'(1,3,4,3)^2 e_3 + (c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2))e_4$. We find that $\alpha(v) \notin \langle \{v\}\rangle$, thus $D_2^2(\mathcal{A})$ is not a Hom-subalgebra of $\mathcal{A}$. Computing $[e_i, e_j, v]$ for all $1 \leq i < j \leq 4$, we find that $D_2^2(\mathcal{A})$ is a weak ideal of $\mathcal{A}$ if and only if $(c'(1,3,4,3)c'(1,3,4,4) + c'(1,3,4,2)) = 0$. That is, its bracket is given by

$$[e_1,e_2,e_4] = e_4$$
$$[e_1,e_3,e_4] = -c'(1,3,4,3)c'(1,3,4,4)e_2 + c'(1,3,4,3)e_3 + c'(1,3,4,4)e_4$$

If we take $\mathbb{K} = \mathbb{C}$, $c'(1,3,4,4) = \pm i$ and $c'(1,3,4,3) = -2$ then we get the following two examples where $D_2^2(\mathcal{A})$ is a weak ideal of $\mathcal{A}$:

$$\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= e_4 \\
[e_1, e_3, e_4] &= 2ie_2 - 2e_3 + ie_4 \\
[e_2, e_3, e_4] &= 0
\end{aligned}
\qquad
\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= e_4 \\
[e_1, e_3, e_4] &= -2ie_2 - 2e_3 - ie_4 \\
[e_2, e_3, e_4] &= 0.
\end{aligned}$$

The end

Thank you !