Graded von Neumann regularity of rings graded by semigroups

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Reference

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Conventions

A ring R will always be associatve but not necessarily commutative or unital.

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We will consider R equipped by group, semigroup and groupoid gradings.

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Von Neumann regular rings

Definition (Von Neumann, 1936)

A unital ring R is called *von Neumann regular* if for each $a \in R$ there is some $x \in R$ such that $a = axa$.

Generalizes to non-unital rings in an obvious manner.

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Proposition

Let R be an s-unital ring. The following are equivalent:

1. R is *von Neumann regular*
2. every finitely generated left (right) ideal is generated by an idempotent

Group graded von Neumann regular rings

Definition

Let R be a ring and let G be a group. A *G-grading* of R is a collection of additive subgroups $\{R_g\}_{g \in G}$ such that $R = \bigoplus_{g \in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. Moreover, R_e is called *the principal component*.

Example

Let R be a unital ring and let G be a group. The group ring $R[G]$ is the free module with basis elements $\{\delta_g \mid g \in G\}$. It becomes a unital ring with multiplication defined by $(a\delta_g)(b\delta_h) = ab\delta_{gh}$ for all $a, b \in R, g, h \in G$.

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Definition (Năstăsescu, Oystaeyen, 1982)

Let $S = \bigoplus_{g\in G} S_g$ be a G-graded ring. If, for each $g \in G$ and every $a \in S_g$, there is some $x \in S$ such that $a = axa$, then S is called *graded von Neumann regular*.

Group graded von Neumann regular rings and strongly group graded rings

Definition

A G-grading $S = \bigoplus_{g \in G} S_g$ is called *strong* if $S_g S_h = S_{gh}$ for all $g, h \in G$.

Group graded von Neumann regular rings and strongly group graded rings

Definition

A G-grading $S = \bigoplus_{g \in G} S_g$ is called *strong* if $S_g S_h = S_{gh}$ for all $g, h \in G$.

Theorem (Năstăsescu, Oystaeyen, 1982)

Let $S = \bigoplus_{g \in G} S_g$ be a unital strongly G-graded ring. Then S is graded von Neumann regular if and only if S_e is von Neumann regular.

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Example

Let R be a von Neumann regular ring. Then the group ring $R[G]$ and the Laurent polynomial ring $R[x, x^{-1}]$ are graded von Neumann regular.

Nearly epsilon-strongly group graded rings

Proposition

If R is an s-unital strongly G-graded ring, then $R_g R_{g^{-1}} = R_e$ is s-unital for every $g \in G$.

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Definition (Nystedt, Öinert, 2019)

Let G be a group and let R be a G-graded ring. If, for every $g \in G$ and $s \in R_g$ there exist some $\epsilon_g(s) \in R_g R_{g^{-1}}, \epsilon_g(s)' \in R_{g^{-1}} R_g$ such that $\epsilon_g(s) s = s = s \epsilon_g(s)'$, then R is called *nearly epsilon-strongly G-graded*.

Nearly epsilon-strongly group graded rings

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Remark

s-unital strongly G-graded $\implies$ nearly epsilon-strongly G-graded

Graded von Neumann regular rings and nearly epsilon-strongly graded rings

Theorem (L., 2019)

Let G be a group and let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring. Then R is graded von Neumann regular if and only if R is nearly epsilon-strongly G-graded and R_e is von Neumann regular.

Graded von Neumann regular rings and nearly epsilon-strongly graded rings

Theorem (L., 2019)

Let G be a group and let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring. Then R is graded von Neumann regular if and only if R is nearly epsilon-strongly G-graded and R_e is von Neumann regular.

Main application:

Theorem (L., 2019)

Let R be a unital ring and let E be a directed graph. Then the Leavitt path algebra $L_R(E)$ is graded von Neumann regular if and only if R is von Neumann regular.

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Definition

A semigroup S is called *regular* if $Q(s) \neq \emptyset$ for every $s \in S$. Equivalently, $V(s) \neq \emptyset$ for every $s \in S$.

Inverse semigroups

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Example

Let A be a unital ring and consider the full matrix ring $M_n(A)$ for some arbitrary $n > 0$. For $i, j \in \{1, ..., n\}$, let $e_{i,j}$ denote the standard matrix unit. Note that $B_n := \{0\} \cup \{e_{i,j} | 1 \leq i \leq n, 1 \leq j \leq n\}$ is an inverse semigroup under matrix multiplication.

Semigroup graded rings

Definition

Let S be a semigroup and let R be a ring. A S-grading is a collection of additive subgroups $\{R_g\}_{g\in G}$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$.

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Example

Let A be a unital ring and let S be a semigroup. The semigroup ring $A[S]$ is canonically S-graded by putting $(A[S])_s := A\delta_s$ for every $s \in S$.

Groupoids

Definition

A groupoid G is a small category where every morphism is invertible. Let $G^{(2)}$ denote the subset of $G \times G$ of composable morphims. (= group with partial operation)

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Example

Given a topological space X. Consider the fundamental groupoid $\pi_1(X)$. The objects are the set X. The morphisms from point p to the point q are equivalent classes (wrt homotopy) of continous paths from p to q. The fundamental groups $\pi_1(X, x)$ are the vertex groups of $\pi_1(X)$.

Groupoid graded rings

Definition

Let G be a groupoid and let R be a ring. A G-grading of R is a collection of additive subgroups $\{R_g\}_{g\in G}$ such that $R = \bigoplus_{g\in G} R_g$ and

$$R_g R_h \subseteq \begin{cases} R_{gh} & (g,h) \in G^{(2)} \\ 0 & (g,h) \notin G^2(2) \end{cases}$$

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Example

Let A be a ring and let G be a groupoid. The groupoid ring $A[G]$ is canonically G-graded by putting $(A[G])_g := A\delta_g$.

Partial skew groupoid rings

Partial skew groupoid rings generalize both partial group rings and groupoid rings.

Definition (Bagio-Flores-Paques, 2010)

Let G be a groupoid and let A be a unital ring. Consider a unital partial action α of G on A. The *partial skew groupoid ring* $A \star_\alpha G$ is the set of finite sums $\sum_{g \in G} a_g \delta_g$ with multiplication defined by

$$(a\delta_g)(b\delta_h) = \begin{cases} a_g \alpha_g(a_h 1_{g^{-1}})\delta_{gh} & (g,h) \in G^{(2)} \\ 0 & \text{otherwise} \end{cases}$$

Inverse semigroups from groupoids

We associate an inverse semigroup $S(G)$ to the groupoid G in the following way. For all $g, h \in G$:

1. $S(G) := \{0\} \cup G$
2. $g \star_{S(G)} h := gh$ if $(g, h) \in G^{(2)}$
3. $g \star_{S(G)} h := 0$ if $(g, h) \notin G^{(2)}$
4. $g \star_{S(G)} 0 := 0$
5. $0 \star_{S(G)} h := 0$

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Graded von Neumann regularity redux

New definition:

Definition (L.-Öinert, 2022)

Let S be a semigroup. A S-graded ring R is said to be *graded von Neumann regular* if for all $s \in S, r \in R_s$ and $t \in V(s)$ there is some $y \in R_t$ such that $r = ryr$.

Graded von Neumann regularity redux

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Remark

In the case when S is a group, this definition coincides with the one used by Nastasescu and van Oystaeyen (1982).

Nearly epsilon-strongly semigroup graded rings

Remark

Let S be a semigroup and let R be a S-graded ring.

1. For each $e \in E(S)$, R_e is a subring of R
2. For all $s \in S$ and $t \in V(s)$, R_s is a $R_{st} - R_{ts}$-bimodule
3. For all $s \in S$ and $t \in V(s)$, $R_s R_t$ is an ideal of R_{st} and $R_t R_s$ is an ideal of R_{ts}.

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Definition (L.-Öinert, 2022)

For all $s \in S, t \in V(s)$ and $r \in R_s$, there exist $\epsilon_{s,t}(r) \in R_s R_t$ and $\epsilon_{s,t}(r)' \in R_t R_s$ such that the equalities $\epsilon_{s,t}(r)r = r = r\epsilon_{s,t}(r)'$ hold.

Necessary condition

Lemma

Let R be an S-graded ring which is graded von Neumann regular. Then the following assertions hold:

1. R_e *is von Neumann regular for every* $e \in E(S)$.
2. R *is nearly epsilon-strongly* S*-graded.*

Necessary condition

Lemma

Let R be an S-graded ring which is graded von Neumann regular. Then the following assertions hold:

1. *R_e is von Neumann regular for every $e \in E(S)$.*
2. *R is nearly epsilon-strongly S-graded.*

Proof

(1): Take $e \in E(S)$ and $r \in R_e$. Using that $e = e^3$, by graded von Neumann regularity, we get that there is some $y \in R_e$ such that $r = ryr$.

Necessary condition

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Proof

(1): Take $e \in E(S)$ and $r \in R_e$. Using that $e = e^3$, by graded von Neumann regularity, we get that there is some $y \in R_e$ such that $r = ryr$.
(2): Take $s \in S, t \in V(s)$ and $r \in R_s$. Using that S is graded von Neumann regular, there is some $y \in R_t$ such that $r = ryr$. Put $\epsilon_{s,t}(r) := ry \in R_s R_t$ and $\epsilon_{s,t}(r)' := yr$.

Main result (I)

Lemma (L.-Öinert, 2022)

Let R be a nearly epsilon-strongly S-graded ring and suppose that R_e is von Neumann regular for every $e \in E(S)$. Then, for all $s \in S, t \in V(s)$ and $r \in R_s$, the left R_{ts}-ideal $R_t s$ is generated by an idempotent.

Main result (I)

Lemma (L.-Öinert, 2022)

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Theorem (L.-Öinert, 2022)

Let S be a semigroup and let R be an S-graded ring. The following assertions are equivalent:

1. *R is graded von Neumann regular*
2. *R is nearly epsilon-strongly S-graded and R_e is von Neumann regular for every $e \in E(e)$.*

Main result (II)

Theorem (L.-Öinert, 2022)

Let S be an inverse semigroup and let R be an S-graded ring. The following assertions are equivalent:

1. *R is graded von Neumann regular*
2. *For all $s \in S$ and $r \in R_s$ there exist $t \in V(s)$ and $y \in R_t$ such that $r = ryr$.*
3. *R is nearly epsilon-strongly S-graded and R_e is von Neumann regular for every $e \in E(e)$.*

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Semigroup rings

Corollary (L.-Öinert, 2022)

Let S be a semigroup and let R be a strongly S-graded ring for which R_e is s-unital for every $e \in E(S)$. Then R is graded von Neumann regular if and only if R_e is von Neumann regular for every $e \in E(S)$.

Semigroup rings

Corollary (L.-Öinert, 2022)

Let S be a semigroup and let R be a strongly S-graded ring for which R_e is s-unital for every $e \in E(S)$. Then R is graded von Neumann regular if and only if R_e is von Neumann regular for every $e \in E(S)$.

Proposition (L.-Öinert, 2022)

Let A be an s-unital ring and let S be a semigroup containing at least one idempotent. Equip the semigroup ring $A[S]$ with the canonical S-grading. Then $A[S]$ is graded von Neumann regular if and only if A is von Neumann regular.

Matrix rings

Example

Let A be a unital ring and consider the full matrix ring $M_n(A)$ for some arbitrary $n > 0$. For $i, j \in \{1, ..., n\}$, let $e_{i,j}$ denote the standard matrix unit. Recall that $B_n := \{0\} \cup \{e_{i,j} | 1 \leq i \leq n, 1 \leq j \leq n\}$.

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Putting $(M_n(A))_0 := 0$ and $(M_n(A))_s := As$ for $s \in B_n \setminus \{0\}$ gives a B_n-grading of $M_n(A)$.

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Putting $(M_n(A))_0 := 0$ and $(M_n(A))_s := As$ for $s \in B_n \setminus \{0\}$ gives a B_n-grading of $M_n(A)$.

Example

Consider $n = 3$ and $M_3(A)$ with its B_3-grading. Note that $B_3 = \{0, e_{1,1}, e_{2,2}, e_{3,3}, e_{1,2}, e_{1,3}, e_{2,1}, e_{2,3}, e_{3,1}, e_{3,2}\}$. We can check that the above B_3-grading on $M_3(A)$ is epsilon-strong. Using our results, we conclude that $M_3(A)$ is graded von Neumann regular if and only if A is von Neumann regular.

Groupoid graded rings

Definition (Nystedt-Öinert-Pinedo, 2020)

Let G be a groupoid and let R be a G-graded ring. Then R is called *nearly epsilon-strongly G-graded* if for all $g \in G$ and $r \in R_g$ there exist $\epsilon_{g,g^{-1}}(r) \in R_g R_{g^{-1}}$ and $\epsilon_{g,g^{-1}}(r) \in R_{g^{-1}} R_g$ such that $\epsilon_{g,g^{-1}}(s)r = r = r\epsilon_{g,g^{-1}}(r)$.

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Reduction:

Proposition (L.-Öinert, 2022)

Let G be a groupoid and let R be a G-graded ring. Simultaneously, consider R as an $S(G)$-graded ring, where $S(G)$ is the semigroup associated with G. Then R is nearly epsilon-strongly G-graded if and only if R is nearly epsilon-strongly $S(G)$-graded.

Groupoid graded rings (II)

Theorem (L.-Öinert, 2022)

Let G be a groupd and let R be a G-graded ring. The following assertions are equal:

1. *R is graded von Neumann regular;*
2. *For all $g \in G$ and $r \in R_g$ there exist $y \in R_{g^{-1}}$ such that $r = ryr$;*
3. *R is nearly epsilon-strongly G-graded and R_e is von Neumann regular for every $e \in ob(G)$.*

Thank you for your attention!