Operator representations of covariance type commutation relations

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Introduction

Consider relations of the form

$$AB = BF(A) \tag{1}$$

for a certain function F satisfying certain conditions, where A, B are elements of an associative algebra over a field (for example, field of complex numbers).

This relation appears in Quantum Mechanics, Wavelet Analysis, and have some connection with Dynamical Systems and for specific spaces it is related to Spectral Theory.

Introduction cont.

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- Some of the main objectives are to find representations of (1) and study their properties.
- We construct representations of Relation (1) by linear integral and multiplication operators on L_p spaces.

Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space.

Proposition

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, *be defined as follows*

$$(Ax)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

for almost every t, *where* $k(t,s) : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, *is a measurable function, and* $b : \mathbb{R} \to \mathbb{R}$ *is a measurable function.*

Consider a polynomial $F(z) = \delta_0 + \delta_1 z + \delta_2 z^2 + \ldots + \delta_n z^n$, where $\delta_0, \delta_1, \ldots, \delta_n$ are real constants. Set

$$k_0(t,s) = k(t,s),\ k_m(t,s) = \int\limits_{\alpha}^{\beta} k(t,\tau) k_{m-1}(\tau,s) d\tau, \quad m = \overline{1,n}$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s). \tag{2}$$

Then, $AB = BF(A)$ if and only if for all $x \in L_p(\mathbb{R})$

$$b(t)\delta_0 x(t) + b(t)\int\limits_{\alpha}^{\beta} F_n(k(t,s))x(s)ds = \int\limits_{\alpha}^{\beta} k(t,s)b(s)x(s)ds. \tag{3}$$

Then, $AB = BF(A)$ if and only if for all $x \in L_p(\mathbb{R})$

$$b(t)\delta_0 x(t) + b(t)\int\limits_{\alpha}^{\beta} F_n(k(t,s))x(s)ds = \int\limits_{\alpha}^{\beta} k(t,s)b(s)x(s)ds. \quad (3)$$

If $\delta_0 = 0$, that is, $F(z) = \delta_1 z + \delta_2 z^2 + \ldots + \delta_n z^n$ then the condition (3) reduces to the following: for almost every (t,s) in $\mathbb{R} \times [\alpha, \beta]$,

$$b(t)F_n(k(t,s)) = k(t,s)b(s). \quad (4)$$

Corollary

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, *be defined as follows*

$$(Ax)(t) = \int\limits_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

for almost every t, *where* $k(t,s) : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, *is a measurable function,* $b \in L_\infty(\mathbb{R})$ *nonzero such that the set*

$$\operatorname{supp} b \cap [\alpha, \beta]$$

has measure zero. Consider $F(z) = \delta_0 + \delta_1 z + \delta_2 z^2 + \ldots + \delta_n z^n$, *where* $\delta_0, \ldots, \delta_n$ *are real constants.*

We set

$$k_0(t,s) = k(t,s),\ k_m(t,s) = \int_{\alpha}^{\beta} k(t,\tau)k_{m-1}(\tau,s)d\tau, \quad m=\overline{1,n}$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s), \quad n \in \mathbb{N}.$$

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Then, we have $AB = BF(A)$ if and only if $\delta_0 = 0$ and the set

$$(\operatorname{supp} b \times [\alpha,\beta]) \cap \operatorname{supp} g_{Fk}$$

has measure zero in $\mathbb{R} \times [\alpha,\beta]$, where $g_{Fk} : \mathbb{R} \times [\alpha,\beta] \to \mathbb{R}$ defined by $g_{Fk}(t,s) = F_n(k(t,s))$.

Corollary

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, *be defined as follows*

$$(Ax)(t) = \int\limits_{\alpha}^{\beta} a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

for almost every t, *where* $a \in L_p(\mathbb{R})$, $c \in L_q([\alpha, \beta])$ $(\alpha, \beta \in \mathbb{R})$, $1 < q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $b \in L_\infty(\mathbb{R})$. *Consider* $F(z) = \delta_1 z + \delta_2 z^2 + \ldots + \delta_n z^n$, *where* $\delta_1, \ldots, \delta_n$ *are real constants. Set*

$$Q = \int\limits_{\alpha}^{\beta} a(s)c(s)ds.$$

Then, we have $AB = BF(A)$ if and only if the set

$$\operatorname{supp} g_{ac} \cap \operatorname{supp} g_b,$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$, where g_{ac}, $g_b : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$ are defined as follows

$$\begin{aligned} g_{ac}(t,s) &= a(t)c(s) \\ g_b(t,s) &= b(t)\sum_{j=1}^{n} \delta_j Q^{j-1} - b(s). \end{aligned}$$

Example 1

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows, for almost all t,

$$(Ax)(t) = \int_0^2 a(t)c(s)x(s)ds, \ (Bx)(t) = b(t)x(t),$$

where $a(t) = 2tI_{[0,2]}(t)$, $c(s) = I_{[0,1]}(s)$, $b(t) = I_{[1,2]}(t)t^2$. Consider $F(z) = \delta_1 z + \delta_2 z^2 + \ldots + \delta_n z^n$, where $\delta_i \in \mathbb{R}$, $i = \overline{1,n}$. Then, the above operators satisfy the relation $AB = BF(A)$ if and only if $\sum_{j=1}^{n} \delta_j = 0$.

Proposition

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int\limits_{\alpha}^{\beta} k(t,s)x(s)ds$$

for almost every t, where $a \in L_\infty(\mathbb{R})$, $k(t,s) : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, is a Lebesgue measurable function. Consider $F(z) = \delta_0 + \delta_1 z + \delta_2 z^2 + \ldots + \delta_n z^n$, where $\delta_0, \delta_1, \ldots, \delta_n$ are constants.

Then

$$AB = BF(A)$$

if and only if the set

$$\operatorname{supp} g_{aF} \cap \operatorname{supp} k$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$, where $g_{aF}(t,s) = a(t) - F(a(s))$.

Example 2

Let $A: L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B: L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds$$

for almost every t, where $a(t) = \gamma_0 + I_{[\alpha, \frac{\alpha+\beta}{2}]}(t)t^2$, γ_0 is a real number, $b(t) = (1+t^2)I_{[\beta+1, \beta+2]}(t)$, $c(s) = I_{[\frac{\alpha+\beta}{2}, \beta]}(s)(1+s^4)$, $\alpha, \beta \in \mathbb{R}$. Let $F(z) = \delta_0 + \delta_1 z$, where $\delta_0, \delta_1 \in \mathbb{R}$ and $\delta_1 \neq 0$. If $\delta_0 = \gamma_0 - \delta_1\gamma_0$ then the above operators satisfy the relation

$$AB - \delta_0 BA = \delta_1 B.$$

Theorem

Let (X, Σ, μ) *be a* σ*-finite measure space. Let* $A : L_p(X,\mu) \to L_p(X,\mu)$, $B : L_p(X,\mu) \to L_p(X,\mu)$, $1 \le p \le \infty$ *be nonzero operators defined as follows, for almost every* t,

$$(Ax)(t) = \int\limits_{G_A} k_A(t,s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{G_B} k_B(t,s)x(s)d\mu_s,$$

where, $G_A,\ G_B \in \Sigma$, $\mu(G_A) < \infty$, $\mu(G_B) < \infty$, $k_A(t,s) : X \times S_A \to \mathbb{R}$, $k_B(t,s) : X \times S_B \to \mathbb{R}$ *are measurable functions. Let* $F(z) = \sum\limits_{j=0}^{n} \delta_j z^j$, $\delta_j \in \mathbb{R}$, $j = \overline{0,n}$. *Set* $G = G_A \cap G_B$,
$k_{0,A}(t,s) = k_A(t,s)$, $k_{m,A}(t,s) = \int\limits_{G_A} k_A(t,\tau)k_{m-1,A}(\tau,s)d\mu_\tau$,
$F_n(k_A(t,s)) = \sum\limits_{j=1}^{n} \delta_j k_{j-1,A}(t,s)$.

Then $AB = BF(A)$ if and only if the following conditions are fulfilled:

1. for almost every $(t,\tau) \in X \times G$,
$$\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s - \delta_0 k_B(t,\tau) = \int\limits_{G_B} k_B(t,s)F_n(k_A(s,\tau))d\mu_s;$$
2. for almost every $(t,\tau) \in X \times (G_B \setminus G)$,
$$\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s = \delta_0 k_B(t,\tau);$$
3. for almost every $(t,\tau) \in X \times (G_A \setminus G)$,
$$\int\limits_{G_B} k_B(t,s)F_n(k(s,\tau))d\mu_s = 0.$$

Corollary

Let (X, Σ, μ) *be a* σ*-finite measure space. Let* $A : L_p(X,\mu) \to L_p(X,\mu)$, $B : L_p(X,\mu) \to L_p(X,\mu)$, $1 \le p \le \infty$ *be nonzero operators defined as follows, for almost every* t,

$$(Ax)(t) = \int\limits_{G_A} k_A(t,s)x(s)d\mu_s, \quad (Bx)(t) = \int\limits_{G_B} k_B(t,s)x(s)d\mu_s,$$

where $G_A,\ G_B \in \Sigma$, $\mu(G_A) < \infty$, $\mu(G_B) < \infty$, $k_A(t,s) : X \times G_A \to \mathbb{R}$, $k_B(t,s) : X \times G_B \to \mathbb{R}$ *are measurable functions. Let* $\delta \in \mathbb{R} \setminus \{0\}$ *and let* d *be a positive integer. Set*

$$G = G_A \cap G_B,$$

$$k_{0,A}(t,s) = k_A(t,s),\ \ k_{m,A}(t,s) = \int\limits_{G_A} k_A(t,\tau)k_{m-1,A}(\tau,s)d\mu_\tau,\ \ m \ge 1.$$

Then

$$AB = \delta BA^d$$

if and only if the following conditions are fulfilled

1. for almost every $(t,\tau) \in X \times G$,
$$\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s = \delta \int\limits_{G_B} k_B(t,s)k_{d-1,A}(s,\tau)d\mu_s.$$
2. for almost every $(t,\tau) \in X \times (G_B \setminus G)$,
$$\int\limits_{G_A} k_A(t,s)k_B(s,\tau)d\mu_s = 0.$$
3. for almost every $(t,\tau) \in X \times (G_A \setminus G)$,
$$\int\limits_{G_B} k_B(t,s)k_{d-1,A}(s,\tau)d\mu_s = 0.$$

Example 1

Let $(\mathbb{R}, \Sigma, \mu)$ be the standard Lebesgue measure space. Let $F(z) = \delta_0 + \delta_1 z$, $\delta_0, \delta_1 \in \mathbb{R}$, $\delta_1 \neq 1$, $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$ and either $\frac{\beta-\alpha}{\pi} \in \mathbb{Z}$ or $\frac{\beta+\alpha}{\pi} \in \mathbb{Z}$ such that $\sigma_1 = \int_\alpha^\beta \sin^2(s)ds \neq 0$, $\sigma_2 = \int_\alpha^\beta \cos^2(s)ds \neq 0$. Define operators as follows

$$(Ax)(t) = I_{[\alpha_1,\beta_1]}(t) \int\limits_\alpha^\beta \left(\frac{\delta_0\theta_1}{\theta_2\sigma_2(1-\delta_1)} \sin t \cos s + \frac{\delta_0}{\sigma_2(1-\delta_1)} \cos t \cos s \right) x(s)ds,$$

$$(Bx)(t) = I_{[\alpha_1,\beta_1]}(t) \int_\alpha^\beta (\theta_1 \sin t \cos s + \theta_2 \cos t \cos s) x(s)ds,$$

for almost every $t \in \mathbb{R}$, where $\alpha_1 \leq \alpha$, $\beta_1 \geq \beta$, $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_2 \neq 0$.

Then operator A and B satisfy $ABx = (\delta_0 B + \delta_1 BA)x$ for all $x \in L_p(\mathbb{R}, \mu)$, $1 \leq p \leq \infty$.

Example 2

Let $\alpha, \beta, \omega \in \mathbb{R}$ such that $\alpha < \beta$, $\omega \neq 0$ and either $\frac{\omega(\beta-\alpha)}{\pi} \in \mathbb{Z}$ or $\frac{\omega(\beta+\alpha)}{\pi} \in \mathbb{Z}$ such that $\sigma_1 = \int_\alpha^\beta \sin^2(\omega s) ds \neq 0$, $\sigma_2 = \int_\alpha^\beta \cos^2(\omega s) ds \neq 0$. Let

$$
\begin{aligned}
(Ax)(t) &= \int\limits_\alpha^\beta I_{[\alpha_1,\beta_1]}(t) \left[\frac{1}{\delta\sigma_2} \cos(\omega t)\cos(\omega s) - \frac{1}{\delta\sigma_1} \sin(\omega t)\sin(\omega s) \right. \\
&\quad \left. + \theta_{A,4} \cos(\omega t)\sin(\omega s) \right] x(s) d\mu_s, \\
(Bx)(t) &= \int\limits_\alpha^\beta I_{[\alpha_1,\beta_1]}(t) \theta_{B,2} \cos(\omega t)\cos(\omega s) x(s) d\mu_s,
\end{aligned}
$$

for almost every t, where $\theta_{A,4}, \theta_{B,2} \in \mathbb{R}$, $\delta \in \mathbb{R} \setminus \{0\}$.

For all $x \in L_p(\mathbb{R}, \mu)$, $1 \le p \le \infty$ we have

$$ABx = \delta BA^2 x.$$

Moreover, for all $x \in L_p(\mathbb{R}, \mu)$, $1 \le p \le \infty$ we have

$$(AB - BA)x(t) = \theta_{A,4}\sigma_2\theta_{B,2} \int_{\alpha}^{\beta} I_{[\alpha_1,\beta_1]}(t)\cos(\omega t)\sin(\omega s)x(s)d\mu_s,$$

for almost every t.

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Thank you!!!