Paradoxicality in groups and rings

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BTH

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Context

- This talk is serving as a preparation for Karl's talk.
- Today, "ring" means "unital ring".
- 1 denotes both an element of $\mathbb{R}^+$ and the identity element of a group.

Outline

1. Paradoxicality in groups
2. Paradoxicality in rings

The Banach-Tarski paradox (1924)

Recall: $E(3)$, the group of Euclidean motions (rigid transformations) in 3 dimensions, contains a free group on two generators.

Paradoxical decompositions

Let G be a group.

Definition

Two sets $A, B \subseteq G$ are *G-equidecomposable*, written $A \sim_G B$, if there are a partition $\{A_1, \ldots, A_n\}$ of A and elements $g_1, \ldots, g_n \in G$ such that $\{g_1 A_1, \ldots, g_n A_n\}$ is a partition of B.

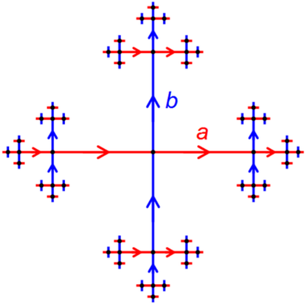
Definition

A subset X of G is said to be *G-paradoxical* if there are sets $A, B \subseteq X$ such that $A \cap B = \emptyset$ and $A \sim_G X \sim_G B$.

Lemma

A subset X of G is G-paradoxical if and only if there is a partition $\{A, B\}$ of X such that $A \sim_G X \sim_G B$.

Example: The free group on two generators, $\mathbb{F}_2 = \langle a, b \rangle$



- Define $W(x) :=$ {all reduced words that start with the letter x}.
- Note that $\mathbb{F}_2 = \{1\} \bigsqcup W(a) \bigsqcup W(a^{-1}) \bigsqcup W(b) \bigsqcup W(b^{-1})$.
- $\mathbb{F}_2 = W(a) \bigsqcup aW(a^{-1}) = W(b) \bigsqcup bW(b^{-1})$

Conclusion: $\mathbb{F}_2$ is $\mathbb{F}_2$-paradoxical.

Amenability

Definition

A group G is *amenable* if there is a function $\mu : \mathcal{P}(G) \to [0,1]$ with the following three properties:

1. $\mu(A \cup B) = \mu(A) + \mu(B)$ for any sets $A, B \subseteq G$ with $A \cap B = \emptyset$;
2. $\mu(gA) = \mu(A)$ for any set $A \subseteq G$;
3. $\mu(G) = 1$.

Remark

If G is amenable, then G is not G-paradoxical.

Examples of amenable groups: Finite groups

Example

Let G be a finite group. Define $\mu : \mathcal{P}(G) \to [0,1]$ by

$$\mu(A) := \frac{|A|}{|G|}.$$

Then

1. $\mu(A \cup B) = \mu(A) + \mu(B)$ for any sets $A, B \subseteq G$ with $A \cap B = \emptyset$;
2. $\mu(gA) = \mu(A)$ for any set $A \subseteq G$, and any $g \in G$;
3. $\mu(G) = 1$.

More examples of amenable groups

Example (Amenable groups)

- All abelian groups.
- All solvable groups.
- Finitely generated groups of subexponential growth.

Remark

- A subgroup of an amenable group is amenable.
- A quotient of an amenable group is amenable.
- A group extension of an amenable group by an amenable group is again amenable.
- A direct limit of amenable groups is amenable.

The Følner condition (1955)

Definition

Let G be a group. We say that G satisfies *the Følner condition* if, for any finite subset K of G and $\epsilon \in \mathbb{R}^+$, there exists a finite subset F of G such that $|KF| < (1+\epsilon)|F|$.

Theorem (Følner, 1955)

Let G be a group. Then the following two statements are equivalent.

1. *G is amenable.*
2. *G satisfies the Følner condition.*

Supramenability

Definition

Let G be a group and let $X \subseteq G$. We say that X is *amenable with respect to G* if there is a function $\mu : \mathcal{P}(G) \to [0, \infty)$ with the following three properties:

1. $\mu(A \cup B) = \mu(A) + \mu(B)$ for any sets $A, B \subseteq G$ with $A \cap B = \emptyset$;
2. $\mu(gA) = \mu(A)$ for any set $A \subseteq G$, and any $g \in G$;
3. $\mu(X) = 1$.

Definition

A group G is *supramenable* if every nonempty subset of G is amenable with respect to G.

Remark

Every supramenable group is amenable.

Examples of supramenable groups

Example (Supramenable groups)

- All abelian groups.
- Any group all of whose finitely generated subgroups display a subexponential rate of growth.
- Every locally virtually nilpotent group.

Remark

The class of supramenable groups is closed under taking subgroups, quotients and direct limits.

Tarski's theorem

Theorem (Tarski, 1929)

Let G be a group and X a subset of G. Then the following two statements are equivalent.

1. *X is G-paradoxical.*
2. *X is not amenable with respect to G.*

Remark

- A group G is amenable if and only if G is not G-paradoxical.
- A group G is supramenable if and only if no nonempty subset of G is G-paradoxical.

The Følner/Rosenblatt condition

Definition (Rosenblatt, 1973)

Let G be a group. If $X \subseteq G$, then we say that G satisfies *the Følner condition with respect to X* if, for any finite subset K of G and $\epsilon \in \mathbb{R}^+$, there exists a finite subset F of G such that $|KF \cap X| < (1+\epsilon)|F \cap X|$.

Theorem

Let G be a group and X a subset of G. Then the following two statements are equivalent.

1. *X is amenable with respect to G.*
2. *G satisfies the Følner condition with respect to X.*

Vector spaces and free modules

Consider the $\mathbb{R}$-vector space $\mathbb{R}^n$ and recall:

- Any generating set for $\mathbb{R}^n$ has cardinality $\leq n$.
- Any linearly independent set in $\mathbb{R}^n$ has cardinality $\leq n$.

Example

Let n, m be positive integers. Suppose that $\mathbb{R}^n \cong \mathbb{R}^m$ as vector spaces (over $\mathbb{R}$). Then, $n = m$.

Free modules

Example

Let R be a commutative ring. Suppose that $R^n \cong R^m$ as R-modules. Then $n = m$.

Definition (Invariant basis number)

A ring R is said to have *IBN* if whenever $R^n \cong R^m$ as R-modules, we must have $n = m$.

Definition (Unbounded generating number)

A ring R is said to have *UGN* if whenever there is an R-module epimorphism $R^n \to R^m$, we must have $n \geq m$.

Remark (Left/right symmetry)

We do not need to distinguish left modules from right modules at this point.

Matrix characterizations of IBN and UGN

Proposition

A ring R does not have UGN if and only if, for some integers $n > k > 1$, there exist an $n \times k$ matrix A and a $k \times n$ matrix B (with coefficients in R) such that $AB = I_n$.

Proposition

A ring R does not have IBN if and only if, for some integers $n > k > 1$, there exist an $n \times k$ matrix A and a $k \times n$ matrix B (with coefficients in R) such that $AB = I_n$ and $BA = I_k$.

Remark

(a) If R has UGN, then R has IBN.
(b) If $f : S \to R$ is a ring morphism, and R has UGN (resp. IBN), then S has UGN (resp. IBN).

Example: The Leavitt algebra $R := L(1,2)$

Consider the following Leavitt algebra:

$$R := \mathbb{C}\langle a, b, a^*, b^* | a^*b = 0, b^*a = 0, a^*a = 1, b^*b = 1, aa^* + bb^* = 1\rangle$$

Define $f : R \to R^2$ by

$$f(r) := (ra, rb)$$

and $g : R^2 \to R$ by

$$g(r_1, r_2) := r_1a^* + r_2b^*.$$

Note that

- $(f \circ g)(r_1, r_2) = f(g(r_1, r_2)) = f(r_1a^* + r_2b^*) = ((r_1a^* + r_2b^*)a, (r_1a^* + r_2b^*)b) = (r_1, r_2)$, and
- $(g \circ f)(r) = g(f(r)) = g(ra, rb) = raa^* + rbb^* = r.$

Conclusion: $R \cong R^2$ as left R-modules.

BGN-rings

Definition/Lemma

Let R be a ring and $n \in \mathbb{Z}^+$. Then the following four statements are equivalent.

1. $\mathrm{gn}(R) = n$, the so-called *generating number of R*.
2. The integer n is the smallest positive integer such that there is an R-module epimorphism $R^n \to R^{n+1}$.
3. The integer n is the smallest positive integer such that there is an R-module epimorphism $R^n \to R^m$ for some integer $m > n$.
4. The integer n is the smallest positive integer such that there is an R-module epimorphism $R^n \to R^m$ for every $m \in \mathbb{Z}^+$.
5. The integer n is the smallest positive integer such that every finitely generated R-module is a homomorphic image of R^n.

The end

THANK YOU FOR YOUR ATTENTION!