Commutation relations for a class of noncommutative spheres

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March 22, 2024

SNAG 2024

The first lore slide

My entire PhD project revolved around the noncommutative geometry of structures called *real calculi*. A real calculus is a structure $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}, M, \varphi)$, where

- $\mathcal{A}$ is a unital *-algebra,
- $\mathfrak{g}$ is a real Lie algebra of hermitian derivations,
- M is a (right) $\mathcal{A}$-module, and
- $\varphi : \mathfrak{g} \to M$ is a $\mathbb{R}$-linear map such that $\varphi(\mathfrak{g})$ generates M.

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We can discuss the notion of Levi-Civita connections ∇ in the framework.

The second lore slide

One can construct a theory of isometric embeddings for real calculi, where $(\phi, \psi, \hat{\psi}) : C_{\mathcal{A}} \to C_{\mathcal{A}'}$ denotes an embedding where $C_{\mathcal{A}}$ takes the role of the ambient space and $C_{\mathcal{A}'}$ takes the role of the embedded space.

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For a basis $\{\delta_1, ..., \delta_k\}$ of $\mathfrak{g}'$, define the mean curvature $H_{\mathcal{A}'} : M \to \mathcal{A}'$ as:

$$H_{\mathcal{A}'}(m) = \phi(h(m, \alpha(\delta_j, \varphi \circ \psi(\delta_j))))(h')^{ij}, \quad m \in M.$$

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- α is the NC version of the second fundamental form; the above formula sees the mean curvature as the trace of the second fundamental form.
- $H_{\mathcal{A}'}(m)$ is independent of the choice of basis $\{\delta_1, ..., \delta_k\}$.

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For this reason (and some others) we decided to look at noncommutative spheres.

The NC 3-sphere

Let $\theta \in \mathbb{R}$. The noncommutative 3-sphere S^3_θ is the unital *-algebra with generators Z_1, Z_1^*, Z_2, Z_2^* subject to the relations

$$\begin{array}{lll} Z_2 Z_1 = q Z_1 Z_2 & Z_2^* Z_1 = \bar{q} Z_1 Z_2^* & Z_2 Z_1^* = \bar{q} Z_1^* Z_2 \\ Z_2^* Z_1^* = q Z_1^* Z_2^* & Z_j^* Z_j = Z_j Z_j^* & Z_1 Z_1^* + Z_2 Z_2^* = \mathbb{1}, \end{array}$$

where $q = e^{i2\pi\theta}$.

The NC 4-sphere

The noncommutative 4-sphere S^4_θ is the unital *-algebra with generators $Z_1, Z_1^*, Z_2, Z_2^*, T$ subject to the relations

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where $T = T^*$ is central, and q is as before.

Embedding the 3-sphere into the 4-sphere

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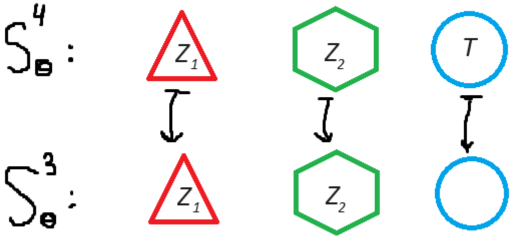
The embedding

By using the principle “square peg $\to$ square hole”, we find $\phi : S^4_\theta \to S^3_\theta$.



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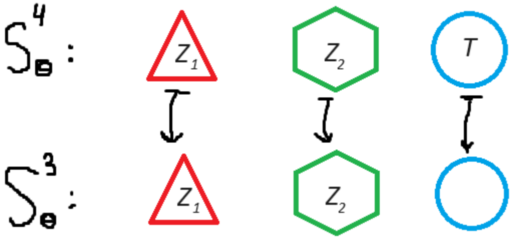

When checking the more technical machinery of embeddings of real calculi, this leads to an embedding of the noncommutative 3-sphere.

The End

Nah, just kidding!

Odd-dimensional NC spheres in general: S_θ^{2n-1} has generators $Z_1, Z_2, ..., Z_n, Z_1^*, ..., Z_n^*$, subject to the relations

$$Z_j Z_i = q_{ij} Z_i Z_j \qquad Z_j^* Z_i^* = q_{ij} Z_i^* Z_j^*$$

$$Z_j^* Z_i = \bar{q}_{ij} Z_i Z_j^* \qquad Z_j Z_i^* = \bar{q}_{ij} Z_i^* Z_j$$

$$Z_j Z_j^* = Z_j^* Z_j = |Z_j|^2,$$

$$|Z_1|^2 + |Z_2|^2 + \cdots + |Z_n|^2 = \mathbb{1}$$

In general, $q_{ij} = e^{i2\pi\theta_{ij}} = \bar{q}_{ji} = e^{-i2\pi\theta_{ji}}$, and $q_{ii} = 1$ for $i, j = 1, ..., n$.

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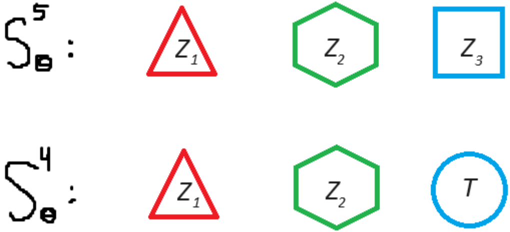
As a special case, consider S_θ^5, with $q_{12} = q = e^{i2\pi\theta}$, which is the same q used for S_θ^3 and S_θ^4.
Can we find a surjective homomorphism $S_\theta^5 \to S_\theta^4$?

A curious observation

We have the following situation.



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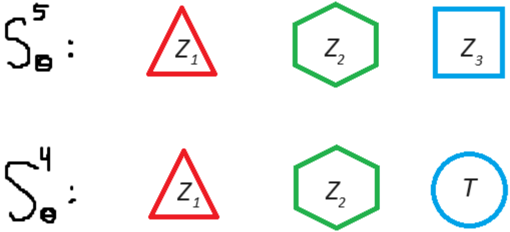

What gives?

Reexamining S^4_θ

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where $T = T^*$ is central.

- Why does T have to be central?
- If T is not central, then what should the commutation relations with Z_1 and Z_2 look like?

Breaking up the generators

The generators Z_1, Z_2, etc. of S_θ^{2n-1} can be seen as noncommutative versions of the complex coordinates $z_1, z_2, \ldots$ for a sphere embedded in $\mathbb{C}^n$, and can therefore be broken up into real and imaginary parts.

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$$X_j = \frac{1}{2}(Z_j + Z_j^*) \quad (\text{"Real part"})$$
$$Y_j = \frac{1}{2}(Z_j - Z_j^*) \quad (\text{"Imaginary part"}),$$

and one can check that $Z_j = X_j + iY_j$, and that $X_j = X_j^*$ and $Y_j = Y_j^*$.

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- Can we express commutation relations in terms of X_j and Y_j?

Commutation relations in terms of X and Y

After doing some computations, one can retrieve the following relations for S_θ^{2n-1} in terms of X_j, Y_j, X_k, Y_k:

$$
\begin{aligned}
X_k X_j &= \mathrm{Re}(q_{jk}) X_j X_k - i\,\mathrm{Im}(q_{jk}) Y_j Y_k, \\
Y_k Y_j &= \mathrm{Re}(q_{jk}) Y_j Y_k - i\,\mathrm{Im}(q_{jk}) X_j X_k, \\
X_k Y_j &= \mathrm{Re}(q_{jk}) X_j Y_k + i\,\mathrm{Im}(q_{jk}) Y_j X_k, \\
Y_k X_j &= \mathrm{Re}(q_{jk}) Y_j X_k + i\,\mathrm{Im}(q_{jk}) X_j Y_k.
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\end{aligned}$$

For S_θ^{2n}, can we assume that T takes the role of X_{n+1}, and that it satisfies the above commutation relations? This would of course lead to some other algebra than what we previously meant by S_θ^{2n}, but maybe such an algebra would work better in the context of embeddings?

A rude awakening

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1. $TX_j = X_j T = 0$ and $\mathrm{Re}(q_{j,n+1})$ or $TY_j = Y_j T = 0$ and $\mathrm{Im}(q_{j,n+1}) = 0$,
2. $TX_j = X_j T = TY_j = Y_j T = 0$, with no restrictions on $q_{j,n+1}$.

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I cannot explain on a conceptual level why it would be essential for T to be a zero divisor. Neither am I certain whether any of the above relations would result in a well-defined structure.

Ditching the complexity

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Instead of using generators Z_j for our algebra, can we work directly with the X_j's themselves. This has some potential advantages, as it would a more unified way of characterizing θ-deformed spheres. Let $\tilde{S}^n_\theta$ denote the unital *-algebra algebra generated by hermitian generators $X_1, ...X_{n+1}$ subject to the relations

$$X_k X_j = q_{jk} X_j X_k, \quad X_1^2 + X_2^2 + ... + X_n^2 + X_{n+1}^2 = \mathbb{1},$$

where $q_{jk} = e^{i2\pi\theta_{jk}}$, and $\theta_{kj} = -\theta_{jk} \in \mathbb{R}$.

Some consequences

Well-definedness nonwithstanding, $\tilde{S}^n_\theta$ may not lead us to the promised land. Let us consider how $\tilde{S}^3_\theta$ and S^3_θ stack up to one another.

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$$Z_2 Z_1 = q Z_1 Z_2 \qquad \tilde{Z}_2 \tilde{Z}_1 = q \tilde{Z}_1 \tilde{Z}_2$$

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$$Z_2^* Z_1 = \bar{q} Z_1 Z_2^* \qquad \tilde{Z}_2^* \tilde{Z}_1 = q \tilde{Z}_1 \tilde{Z}_2^*$$

In other words, the commutation relations on $\tilde{S}^3_\theta$ do not consider its *-structure in a similar way as they do for S^3_θ. While not completely unexpected, it makes me question whether $\tilde{S}^n_\theta$ is the way to go...

The End