On periodic algebras

SNAG workshop 240322

Throughout: A: finite-dimensional k-algebra, k: algebraically closed field.

mod$A$: category of finite-dimensional (right) A-modules.

On periodic algebras

SNAG workshop 240322

Throughout: A: finite-dimensional k-algebra, k: algebraically closed field.

mod$A$: category of finite-dimensional (right) A-modules.

Syzygies: $\Omega(M) \rightarrowtail \underset{\substack{\text{proj.}\\ \text{cover}}}{P} \twoheadrightarrow M \qquad M \in \operatorname{mod} A.$

Minimal proj. resolution:

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] & & P_2 \arrow[rr] \arrow[dr, two heads] & & P_1 \arrow[rr] \arrow[dr, two heads] & & P_0 \arrow[r] & M \arrow[r] & 0 \\
& \Omega^3(M) \arrow[ur, tail] & & \Omega^2(M) \arrow[ur, tail] & & \Omega(M) \arrow[ur, tail] & & &
\end{tikzcd}$$

On periodic algebras

SNAG workshop 240322

Throughout: A: finite-dimensional k-algebra, k: algebraically closed field.

mod$A$: category of finite-dimensional (right) A-modules.

Syzygies: $\Omega(M) \rightarrowtail P \twoheadrightarrow M$ $\quad M \in \mathrm{mod} A$.

proj. cover

Minimal proj. resolution:

$$
\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[dr, twoheadrightarrow] & P_1 \arrow[r] \arrow[dr, twoheadrightarrow] & P_0 \arrow[r] & M \arrow[r] & 0 \\
\Omega^3(M) \arrow[ur, rightarrowtail] & & \Omega^2(M) \arrow[u, rightarrowtail] & \Omega(M) \arrow[u, rightarrowtail] & &
\end{tikzcd}
$$

Def: · $M \in \mathrm{mod} A$ is <u>periodic</u> if $\Omega^n(M) \simeq M$ for some $n \geq 1$

· mod$A$ is <u>periodic</u> if every non-projective $M \in \mathrm{mod} A$ is periodic.

· A is <u>periodic</u> if $A \in \mathrm{mod} A^e$ is periodic

$\left\{ \begin{array}{l} A^e := A^{op} \otimes_k A \\ A \text{ per. as an } A\text{-}A\text{-bimod.} \end{array} \right.$

Some results:

- $M \otimes_A \Omega_{A^e}(A) \simeq \Omega_A(M) \oplus (\text{projectives})$

$\rightsquigarrow$ A periodic $\Longrightarrow$ modA periodic

Some results:

- $M \otimes_A \Omega_{A^e}(A) \simeq \Omega_A(M) \oplus$ (projectives)

$\rightsquigarrow$ A periodic $\Longrightarrow$ mod$A$ periodic

- [Green–Snashall–Solberg '03]: TFAE:

   a) mod$A$ is periodic

   b) all simple A-modules are periodic

   c) A is twisted periodic: $\Omega^n_{A^e}(A) \simeq {}_1A_\sigma$ for some $n \geq 1$, $\sigma \in \mathrm{Aut}_k(A)$

   d) A is self-injective and $\Omega^n \simeq \sigma^*: \underline{\mathrm{mod}}\,A \longrightarrow \underline{\mathrm{mod}}\,A$ for some $n \geq 1$, $\sigma \in \mathrm{Aut}_k(A)$

twisted bimodule:
$a \cdot x \cdot b := a x \sigma(b)$

$\boxed{\begin{array}{l} \mathrm{proj}\,A = \mathrm{inj}\,A \\ \rightsquigarrow \Omega : \underline{\mathrm{mod}}\,A \xrightarrow{\sim} \underline{\mathrm{mod}}\,A \end{array}}$

Some results:

- $M \underset{A}{\otimes} \Omega_{A^e}(A) \simeq \Omega_A(M) \oplus (\text{projectives})$

$\rightsquigarrow$ A periodic $\Longrightarrow$ mod A periodic

- [Green-Snashall-Solberg '03]: TFAE:

  a) mod A is periodic

  b) all simple A-modules are periodic

  c) A is twisted periodic: $\Omega^n_{A^e}(A) \simeq {}_1A_\sigma$ for some $n \geq 1$, $\sigma \in \mathrm{Aut}_k(A)$

  twisted bimodule: $a \cdot x \cdot b := ax\sigma(b)$

  d) A is self-injective and $\Omega^n \simeq \sigma^* : \underline{\mathrm{mod}}\, A \longrightarrow \underline{\mathrm{mod}}\, A$ for some $n \geq 1$, $\sigma \in \mathrm{Aut}_k(A)$

$$\boxed{\begin{array}{l} \mathrm{proj}\, A = \mathrm{inj}\, A \\ \rightsquigarrow \Omega : \underline{\mathrm{mod}}\, A \xrightarrow{\sim} \underline{\mathrm{mod}}\, A \end{array}}$$

- [Carlson '77, GSS'03]: A periodic $\Longrightarrow HH^*(A) / \mathcal{N} \simeq K[x]$

  $\mathcal{N} = \langle \text{homog. nilpotent elements} \rangle \triangleleft HH^*(A)$

A periodic $\Longrightarrow \Omega^n \simeq \mathbb{1}_{\underline{\mathrm{mod}}A}$ for some $n \geq 1$

$$\Downarrow$$

A twisted periodic $\Longleftrightarrow$ simple periodic $\Longleftrightarrow$ modA periodic

$\Longleftrightarrow \exists n, \sigma : \Omega^n \simeq \sigma^*$

$$A \text{ periodic} \Longrightarrow \Omega^n \simeq 1\!\!1_{\underline{\mathrm{mod}}A} \text{ for some } n \geq 1$$

$$\Big\Downarrow$$

$$A \text{ twisted periodic} \Longleftrightarrow \text{simple periodic} \Longleftrightarrow \mathrm{mod}A \text{ periodic}$$

$$\Longleftrightarrow \exists n, \sigma : \Omega^n \simeq \sigma^* \Longleftrightarrow \forall M \in \mathrm{mod}\Lambda : \{\dim_k \Omega^n(M)\}_{n \in \mathbb{N}} \text{ bounded.}$$

[Dugas'12]

A periodic $\Longrightarrow \Omega^n \simeq 1\!\!1_{\mathrm{mod} A}$ for some $n \geq 1$

$\Downarrow$

A twisted periodic $\Longleftrightarrow$ simple periodic $\Longleftrightarrow$ $\mathrm{mod}\,\Lambda$ periodic

$\Longleftrightarrow \exists n, \sigma : \Omega^n \simeq \sigma^* \Longleftrightarrow \forall M \in \mathrm{mod}\,\Lambda : \{\dim_k \Omega^n(M)\}_{n \in \mathbb{N}}$ bounded.

[Dugas'12]

<u>Examples:</u>

- Self-injective algebras of finite representation type [Dugas'10]
- Preprojective algebras of Dynkin quivers
- G: finite group, $p = \mathrm{char}\,k$. Then

  kG: periodic $\Longleftrightarrow$ Sylow p-subgroups of G are cyclic $\Longleftrightarrow$ repr. finite

  or generalised quaternion, $p=2$
- All known examples of self-inj. alg. with a d-cluster-tilting module

Periodicity conjecture [Erdmann-Skowroński '08]

$$\begin{tikzcd}
A \text{ periodic} \arrow[d, Rightarrow] \\
\Omega^n \simeq \mathbb{1}_{\underline{\mathrm{mod}}A} \quad \exists n \geq 1 \arrow[d, Rightarrow] \\
A \text{ twisted periodic} \arrow[uu, Rightarrow, bend left, "?"]
\end{tikzcd}$$

- True for finite group algebras [ES'08]

<u>Trivial extension</u> of A: $T(A) := A \oplus DA$

$\overset{\shortparallel}{}$ $\mathrm{Hom}_k(A,k) \in \mathrm{mod}(A^e)$

with multiplication $(a,f)\cdot(b,g) := (ab, ag+fb)$

$T(A)$ is <u>symmetric</u>: $D(T(A)) \simeq T(A)$ as a bimodule

$\Longrightarrow$ self-injective

<u>Trivial extension</u> of A: $T(A) := A \oplus DA$

$DA = \mathrm{Hom}_k(A,k) \in \mathrm{mod}(A^e)$

with multiplication $(a,f)\cdot(b,g) := (ab, ag+fb)$

$T(A)$ is <u>symmetric</u>: $D(T(A)) \simeq T(A)$ as a bimodule

$\Longrightarrow$ self-injective

<u>Happel's theorem</u>: If $\mathrm{gldim} A < \infty$ then $\mathcal{D}^b(\mathrm{mod} A) \simeq \underline{\mathrm{mod}}^{\mathbb{Z}} T(A)$

$$\mathcal{D}^b(\mathrm{mod} A) \simeq \underline{\mathrm{mod}}^{\mathbb{Z}} T(A) \xrightarrow{\text{forget}} \underline{\mathrm{mod}}\, T(A)$$

Let $\operatorname{gldim} A < \infty$.

Then $\nu = - \overset{\mathbb{L}}{\underset{A}{\otimes}} DA : \mathcal{D}^b(\operatorname{mod} A) \xrightarrow{\sim} \mathcal{D}^b(\operatorname{mod} A)$ is a (the) Serre functor:

$$\operatorname{Hom}_{\mathcal{D}}(X, \nu(Y)) \simeq D \operatorname{Hom}_{\mathcal{D}}(Y, X) \quad \text{(bifunctorially)}$$

Def: A is fractionally Calabi-Yau if $\nu^a \simeq [b]$ for some $a \in \mathbb{Z}$, $b \neq 0$.

(b-Calabi-Yau if $a = 1$
$[b] \simeq \nu$)

__Theorem__ [Chan-D-Iyama-Marczinzik]

1) $\mathcal{T}(A)$ periodic $\iff$ gldim $A < \infty$ and A fractionally Calabi-Yau

2) $\mathcal{T}(A)$ twisted periodic

$\iff$ gldim $A < \infty$ and A twisted fractionally Calabi-Yau:

$\boxed{\nu^a \simeq [b] \circ \varphi^* \text{ for some } a,b; \; \varphi \in \mathrm{Aut}_k(A)}$

$\rightsquigarrow$ Many new examples of periodic and fractionally Calabi-Yau alg.

Ex: P_n: Boolean lattice with 2^n elements.

Then $\mathcal{T}(k[P_n])$ is periodic

with min. period $\begin{cases} 2(n+3) & \text{if } n \text{ even, char}\,k \neq 2, \\ n+3 & \text{else.} \end{cases}$

Ex: P_n: Boolean lattice with 2^n elements.

Then $T(k[P_n])$ is periodic

with min. period $\begin{cases} 2(n+3) & \text{if } n \text{ even, char} k \neq 2, \\ n+3 & \text{else.} \end{cases}$

Corollary: P poset with some $x \in P$ comparable with all other elements.

Then $T(k[P])$ is periodic iff it is twisted periodic.

Ex: P_n: Boolean lattice with 2^n elements.

Then $T(k[P_n])$ is periodic

with min. period $\begin{cases} 2(n+3) & \text{if } n \text{ even, char } k \neq 2, \\ n+3 & \text{else.} \end{cases}$

Corollary: P poset with some $x \in P$ comparable with all other elements.

Then $T(k[P])$ is periodic iff it is twisted periodic.

- What about the periodicity conjecture?
- Does A twisted periodic imply $HH^1(A) = 0$?