A Lie-type construction based on twisted derivations

SNAG 6 workshop

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Structure

References

This talk is based on the following papers:

- (Filippov) V.T. Filippov: *n*-Lie algebras, Siberian Math. J. **26**, 879-891 (1985). Translated from Russian: Sib. Mat. Zh. **26**, 126-140 (1985)
- (GKS) G.García Butenegro, A. Kitouni, S. Silvestrov: On Lie-type constructions over twisted derivations. Non-commutative and Non-associative Algebra and Analysis Structures. SPAS 2019 Springer Proceedings in Mathematics and Statistics, vol. 426 (2023)

n-Lie algebras

n-Lie algebras

An n-Lie algebra as an algebra $\mathcal{A}$ with a totally skew-symmetric n-ary operation $[x_1, \ldots, x_n]$ verifying an ***n*-Jacobi identity**:

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

Derivations of n-Lie algebras

A derivation D of an n-Lie algebra is a $\mathbb{F}$-linear map of $\mathcal{A}$ verifying an n-ary **Leibniz rule**:

$$[x_1, \ldots, x_n]D = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, x_i D, x_{i+1}, \ldots, x_n]$$

Adjoint multiplication on n-Lie algebras

One important characterization of Lie algebras is given by the Jacobi identity: *the adjoint multiplication operator is a derivation of the algebra.*

Adjoint of an n-Lie algebra

The **adjoint operator** of an n-Lie algebra $(\mathcal{A}, [\star, \dots, \star])$ is the linear operator $[\star, y_2, \dots, y_n] : x \longmapsto [x, y_2, \dots, y_n]$

Proposition

The adjoint operator is a derivation of $(\mathcal{A}, [\star, \dots, \star])$.

- This property will be essential to finding a generalization of these structures.

Notation

Across this section Filippov's notation will be used. Filippov applies operators on the right instead of the left:

	Usual notation	Filippov's notation
Composition of maps	$D \circ \sigma$	D
Image by maps	$D(x)$	xD
Image by multiple maps	$D(\sigma(x))$	$x\sigma D$

Table: Usual and Filippov's notations

Across this section we use the following products:

- The dot "$\cdot$" is the commutative associative product on $\mathcal{A}$.
- $[\star, \ldots, \star]$ is an n-ary product on $\mathcal{A}$ (usually the Jacobian).

The Jacobian determiannts

Jacobian of a 2-variable function

The Jacobian determinant of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is the determinant of the Jacobian matrix of partial derivatives.

$$[f_1, f_2] := \left| \frac{df(x,y)}{d(x,y)} \right| = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x} \cdot \frac{\partial f_1}{\partial y}$$

Jacobian of a commutative associative algebra

Given $\{D_1, \ldots, D_n\}$ pairwise commuting derivations of $(\mathcal{A}, \cdot)$, the Jacobian product is defined by

$$[x_1, \ldots, x_n] = |x_i D_j| = \begin{vmatrix} x_1 D_1 & \ldots & x_1 D_n \\ \vdots & \ddots & \vdots \\ x_n D_1 & \ldots & x_n D_n \end{vmatrix}.$$

The Jacobian determinants

This product is skew-symmetric, and derivations verify the following properties:

- The adjoint $[\star, y_2, \ldots, y_n] : x \mapsto [x, y_2, \ldots, y_n]$ is a derivation on $(\mathcal{A}, \cdot)$.
- A derivation D of $(\mathcal{A}, \cdot)$ that commutes with all D_i is a derivation in $(\mathcal{A}, [\star, \ldots, \star])$.

Are these two properties enough to ensure that the adjont is a derivation of $(\mathcal{A}, [\star, \ldots, \star])$?

- Unfortunately not, a bit more artillery is needed.

The n-Lie algebra of Jacobians

Proposition (Filippov, p.576)

For any two square matrices $A = (a_{ij})$ and $B = (b_{ij})$ of order n:

$$\sum_{i=1}^{n} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{i-11} & \dots & a_{i-1n} \\ b_{i1} & \dots & b_{in} \\ a_{i+11} & \dots & a_{i+1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} \begin{vmatrix} a_{11} & \dots & a_{1j-1} & b_{1j} & a_{1j+1} & \dots & a_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj-1} & b_{nj} & a_{nj+1} & \dots & a_{nn} \end{vmatrix}.$$

Constructing the Jacobian algebra

Using this result, one can compare both terms of the Jacobi identity:

$$[[x_1,\dots,x_n],y_2,\dots,y_n]-\sum_{i=1}^{n}[x_1,\dots,x_{i-1},[x_i,y_2,\dots,y_n],x_{i+1},\dots,x_n]$$
$$=\sum_{j=1}^{n}\begin{vmatrix} x_1D_1 & \dots & x_1D_{j-1} & \Delta_{1j} & x_1D_{j+1} & \dots & x_1D_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_nD_1 & \dots & x_nD_{j-1} & \Delta_{1j} & x_nD_{j+1} & \dots & x_nD_n, \end{vmatrix}$$

where the Δ_{ij} are certain determinants depending on $x_i, y_2, \dots, y_n$. By taking minors on the Δ_{ij}, we can express this difference as

Constructing the Jacobian algebra

$$= \sum_{j,k=1}^{n} (-1)^{s+k} y_s D_k D_j \begin{vmatrix} x_1 D_1 & \dots & x_1 D_{j-1} & M_{1k} & x_1 D_{j+1} & \dots & x_1 D_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n D_1 & \dots & x_n D_{j-1} & M_{nk} & x_n D_{j+1} & \dots & x_n D_n \end{vmatrix},$$

$$= \sum_{j,k=1}^{n} (-1)^{s+k} y_s D_k D_j \begin{vmatrix} x^{(1)} & \dots & x^{(j-1)} & M_k & x^{(j+1)} & \dots & x^{(n)} \end{vmatrix} = 0.$$

- For each k, j, the terms in $D_j D_k$ and $D_k D_j$ cancel by commutation of the D_i. That is, the difference

$$[[x_1, \dots, x_n], y_2, \dots, y_n] - \sum_{i=1}^{n} [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n]$$

is a sum of zeroes!

The Jacobian algebra

Theorem (Filippov, Theorem 1)

Let $(\mathcal{A}, \cdot)$ be a commutative associative algebra, $\{D_1, \ldots, D_n\}$ pairwise commuting derivations of $(\mathcal{A}, \cdot)$. The Jacobian algebra $(\mathcal{A}, [\star, \ldots, \star])$, where $[x_1, \ldots, x_n] = |x_i D_j|$, is an n-Lie algebra.

There are two important aspects to consider on this construction:

- The derivatives $\{D_1, \ldots, D_n\}$ commute.
- The derivatives $\{D_1, \ldots, D_n\}$ are *untwisted.*

A proper generalization entails...

The Jacobian product is a product of images of elements by derivations of the algebra. In order to find a proper generalization of Filippov's Jacobian algebra one needs to have:

- operators generalizing derivations ($\rightsquigarrow$ (σ, τ)-derivations),
- a product generalizing the Jacobian ($\rightsquigarrow$ totally skew-symmetric),
- a modified version of the Jacobi identity ($\rightsquigarrow$ hom-Lie or hom-Leibniz),
- one or more twisting maps, if we go the hom-algebra route.

n-hom-Lie algebras

n-hom-Lie algebras

An n-hom-Lie algebra is an hom-algebra $(\mathcal{A}, [\star, \dots, \star], \alpha)$ where α is a linear map on $\mathcal{A}$, $[\star, \dots, \star]$ is totally skew-symmetric and the n-**hom-Jacobi identity** holds:

$$[[x_1, \dots, x_n], y_2\alpha, \dots, y_n\alpha] = \sum_{i=1}^{n} [x_1\alpha, \dots, x_{i-1}\alpha, [x_i, y_2, \dots, y_n], x_{i+1}\alpha, \dots, x_n\alpha]$$

In these algebras the adjoint multiplication is not a derivation, but if $y_2, \dots, y_n$ are fixed points of α it obeys a Leibniz-type rule twisted by α.

Twisted derivations

(σ, τ)-derivations on n-ary algebras

A (σ, τ)-derivation D on an n-ary algebra $(\mathcal{A}, [\star, \dots, \star])$ is a linear operator on $\mathcal{A}$ verifying the twisted n-ary **Leibniz rule**, for all $x_1, \dots, x_n \in \mathcal{A}$,

$$[x_1, \dots, x_n]D = \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, x_i D, x_{i+1}\tau, \dots, x_n\tau]$$

If $n = 2$, this condition is the twisted **Leibniz rule**:

$$(x \cdot y)D = xD \cdot y\tau + x\sigma \cdot yD.$$

We can use this operators to define a new operator generalizing the Jacobian.

Generalized Jacobian

Generalized Jacobian of n elements

Let $\{D_i : i = 1, \ldots, n\}$ be (σ_i, τ_i)-derivations of $(\mathcal{A}, \cdot)$. The **generalized Jacobian** of n elements is the determinant

$$[x_1, \ldots, x_n]_g = \begin{vmatrix} x_1 D_1 & \ldots & x_1 D_n \\ \vdots & \ddots & \vdots \\ x_n D_1 & \ldots & x_n D_n \end{vmatrix}.$$

- This determinant is totally skew-symmetric.

Properties of the generalized Jacobian

The adjoint map being a derivation of the algebra characterizes Lie algebras. The generalized Jacobian (in general) does not verify this.

- If the D_i are pairwise commuting (σ, τ)-derivations commuting with σ and τ, familiar relations are obtained.

Proposition

Let $y_2, \ldots, y_n \in \mathcal{A}$. The linear operator $D : x \mapsto [x, y_2, \ldots, y_n]_g$ is a (σ, τ)-derivation on $(\mathcal{A}, \cdot)$.

Proposition

Let D be a (σ, τ)-derivation on $(\mathcal{A}, \cdot)$ such that $DD_i = D_i D$ for all i. Then D is a (σ, τ)-derivation on $(\mathcal{A}, [\star, \ldots, \star]_g)$.

- Observe that D need not commute with σ and τ.
- These properties indicate that one can open nested generalized Jacobians using the corresponding Leibniz-type rule.

Constructing a Jacobian hom-algebra

For

$$[[x_1,\dots,x_n]_g, y_2\tau,\dots,y_n\tau]_g - \sum_{i=1}^{n}[x_1\sigma,\dots,x_{i-1}\sigma,[x_i,y_2,\dots,y_n]_g,x_{i+1}\tau,\dots,x_n\tau]_g$$

$$= \sum_{i=1}^{n}\begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ x_{i-1}\sigma D_1 & \dots & x_{i-1}\sigma D_n \\ \Delta_{i1} & \dots & \Delta_{in} \\ x_{i+1}\tau D_1 & \dots & x_{i+1}\tau D_n \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix} (= \Delta_s)$$

where $\Delta_{ij} = \sum_{s=2}^{n}[x_i\sigma, y_2\sigma,\dots,y_{s-1}\sigma, y_s D_j, y_{s+1}\tau,\dots,y_n\tau]_g$.

Constructing a Jacobian hom-algebra

One may be tempted to use Filippov's trick, but in general it is not possible.

- The amount of iterations of σ and τ is variable!

Consider first $\sigma = \tau$, that is, the D_i are all (σ, σ)-derivations. Here the difference above takes the form

$$\sum_{j=1}^{n} \begin{vmatrix} x_1 \sigma D_1 & \dots & x_1 \sigma D_{j-1} & \Delta_{1j} & x_1 \sigma D_{j+1} & \dots & x_1 \sigma D_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n \sigma D_1 & \dots & x_n \sigma D_{j-1} & \Delta_{1j} & x_n \sigma D_{j+1} & \dots & x_n \sigma D_n \end{vmatrix}.$$

We can apply Filippov's trick, once again take minors over the column with $D_k D_j$ and obtain a sum of zeroes.

The n-hom-Lie Jacobian algebra

Theorem (GKS, Theorem 10)

Let $(\mathcal{A}, \cdot)$ be a commutative associative algebra, $\{D_1, \dots, D_n\}$ pairwise commuting (σ, σ)-derivations which commute with σ, $[\star, \dots, \star]_g$ the generalized Jacobian. The triple $(\mathcal{A}, [\star, \dots, \star]_g, \sigma)$, is an n-hom-Lie algebra with n-hom-Jacobi identity

$$[[x_1, \dots, x_n]_g, y_2\sigma, \dots, y_n\sigma]_g = \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, [x_i, y_2, \dots, y_n]_g, x_{i+1}\sigma, \dots, x_n\sigma]_g$$

This construction can take, for example, symmetric (σ, τ)-derivations, and due to commutation they are also symmetric in $(\mathcal{A}, [\star, \dots, \star]_g, \sigma)$.

Are we twisting the Jacobian algebra?

We can look at this construction as a *twist* of Filippov's Jacobian algebra. Let σ be multiplicative and let $D_1, \dots, D_n$ be pairwise commuting derivations.

Twisting the Jacobian algebra

For every D_i, the map $D_i\sigma : x \mapsto xD_i\sigma$ is a (σ, σ)-derivation. By multiplicativity of σ, the generalized Jacobian becomes $[\star, \dots, \star]_g = [\star, \dots, \star]\sigma$.

Since σ commutes with all D_i, taking $D_i\sigma$ or σD_i (which is another (σ, σ)-derivation even if D_i and σ do not commute) makes no difference.

- This construction is, in this case, the Yau twist of the Jacobian algebra.

The case $\sigma \neq \tau$

So far, we have obtained a familiar structure (a twist, even) in exchange for heavy commutation relations.In the case $\sigma \neq \tau$, canceling the difference Δ_s gives rise to a new family of algebras which generalize the idea that *the adjoint is a* (σ, τ)*-derivation*, similarly to how n-hom-Lie algebras do.

(σ, τ, n)-hom-Lie algebras

A (σ, τ, n)-hom-Lie algebra is a quadruple $(\mathcal{A}, [\star, \dots, \star], \sigma, \tau)$, where $[\star, \dots, \star]$ is an n-ary totally skew-symmetric product, σ, τ linear maps on $\mathcal{A}$ and an n-ary twisted **Jacobi identity** holds:

$$[[x_1, \dots, x_n], y_2\tau, \dots, y_n\tau] = \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, [x_i, y_2, \dots, y_n], x_{i+1}\tau, \dots, x_n\tau]$$

- **Note:** these are not n-ary hom-Nambu-Lie algebras, but an entirely new family altogether.

General commutation

Finding conditions that cancel the Δ_s has proven to be a cumbersome task, so this work has gone in an unexpected direction: create the most cumbersome, general statement possible under the most general commutation relations one can find. Considering the following commutation relations, with $\lambda_i, \gamma_{jk} \in \mathcal{A}$:

$$D_k D_j = D_j D_k \cdot \gamma_{jk}, \quad D_i \sigma = \sigma D_i \cdot \lambda_i, \quad D_i \tau = \tau D_i \cdot \lambda_i$$

These, naturally, provide different Leibniz-type rules for the D_i. For example, if $\gamma_{jk} = -1 = \lambda_i \ \forall i, j, k$

$$[x_1, \dots, x_n]_g D_j = \sum_{i=1}^{n} \Big([x_1\sigma, \dots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \dots, x_n\tau]_g \cdot (-1)^{n-1} + x_i D_j D_j [x_1\sigma, \dots, x_{i-1}\sigma, x_{i+1}\tau, \dots, x_n\tau]_g^{(j)} \cdot 2(-1)^{i+j+n-1}. \Big)$$

where the exponent $^{(j)}$ indicates that we take all (σ, τ)-derivations **except** D_j.

General Leibniz-type rules

And more generally,

$$[x_1,\dots,x_n]_g D_j =$$

$$\sum_{i=1}^{n}\begin{vmatrix}
x_1\sigma D_1\cdot\lambda_1 & \dots & x_1\sigma D_{j-1}\cdot\lambda_{j-1} & x_1\sigma D_j\cdot\lambda_j & x_1\sigma D_{j+1}\cdot\lambda_{j+1} & \dots & x_1\sigma D_n\cdot\lambda_n\\
\vdots & & \vdots & \vdots & \vdots & & \vdots\\
x_{i-1}\sigma D_1\cdot\lambda_1 & \dots & x_{i-1}\sigma D_{j-1}\cdot\lambda_{j-1} & x_{i-1}\sigma D_j\cdot\lambda_j & x_{i-1}\sigma D_{j+1}\cdot\lambda_{j+1} & \dots & x_{i-1}\sigma D_n\cdot\lambda_n\\
x_i D_j D_1\cdot\lambda_1 & \dots & x_i D_j D_{j-1}\cdot\lambda_{j-1} & x_i D_j D_j & x_i D_j D_{j+1}\cdot\lambda_{j+1}^{-1} & \dots & x_i D_j D_n\cdot\lambda_n^{-1}\\
x_{i+1}\tau D_1\cdot\lambda_1 & \dots & x_{i+1}\tau D_{j-1}\cdot\lambda_{j-1} & x_{i+1}\tau D_j\cdot\lambda_j & x_{i+1}\tau D_{j+1}\cdot\lambda_{j+1} & \dots & x_{i+1}\tau D_n\cdot\lambda_n\\
\vdots & & \vdots & \vdots & \vdots & & \vdots\\
x_n\tau D_1\cdot\lambda_1 & \dots & x_n\tau D_{j-1}\cdot\lambda_{j-1} & x_n\tau D_j\cdot\lambda_j & x_n\tau D_{j+1}\cdot\lambda_{j+1} & \dots & x_n\tau D_n\cdot\lambda_n
\end{vmatrix}$$

$$= \sum_{i=1}^{n}\Delta_{ij}$$

General Leibniz-type rules

In terms of the generalized Jacobian, each Δ_{ij} looks like

$$\Delta_{ij} = \sum_{k=1}^{j-1} \left(x_i D_j D_k \lambda_k (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]_g^{(k)} \prod_{s\neq k} \lambda_s \right)$$
$$+ x_i D_j D_j (-1)^{i+j} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]_g^{(j)} \prod_{s\neq j} \lambda_s$$
$$+ \sum_{k=j+1}^{n} \left(x_i D_j D_k \lambda_k^{-1} (-1)^{i+k} \cdot [x_1\sigma, \ldots, x_{i-1}\sigma, x_{i+1}\tau, \ldots, x_n\tau]_g^{(k)} \prod_{s\neq k} \lambda_s \right).$$

provided that λ_k is invertible, $1 \leq k \leq j-1$. From now on, consider the commutation constants to be invertible.

General Leibniz-type rules

$$[x_1,\dots,x_n]_g D_j = \sum_{i=1}^{n} \begin{vmatrix} x_1\sigma D_1\lambda_1 & \dots & x_1\sigma D_{j-1}\lambda_{j-1} & x_1\sigma D_j\lambda_j & x_1\sigma D_{j+1}\lambda_{j+1} & \dots & x_1\sigma D_n\lambda_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_{i-1}\sigma D_1\lambda_1 & \dots & x_{i-1}\sigma D_{j-1}\lambda_{j-1} & x_{i-1}\sigma D_j\lambda_j & x_{i-1}\sigma D_{j+1}\lambda_{j+1} & \dots & x_{i-1}\sigma D_n\lambda_n \\ x_i D_j D_1\gamma_{j1} & \dots & x_i D_j D_{j-1}\gamma_{jj-1} & x_i D_j D_j & x_i D_j D_j\gamma_{jj+1} & \dots & x_i D_j D_n\gamma_{jn} \\ x_{i+1}\tau D_1\lambda_1 & \dots & x_{i+1}\tau D_{j-1}\lambda_{j-1} & x_{i+1}\tau D_j\lambda_j & x_{i+1}\tau D_{j+1}\lambda_{j+1} & \dots & x_{i+1}\tau D_n\lambda_n \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_n\tau D_1\lambda_1 & \dots & x_n\tau D_{j-1}\lambda_{j-1} & x_n\tau D_j\lambda_j & x_n\tau D_{j+1}\lambda_{j+1} & \dots & x_n\tau D_n\lambda_n \end{vmatrix}$$

$$= \sum_{i=1}^{n} \Delta_{ij} = \sum_{k=1}^{n} x_i D_j D_k (-1)^{i+k} \cdot [x_1\sigma,\dots,x_{i-1}\sigma, x_{i+1}\tau,\dots,x_n\tau]_g^{(k)} \left(\gamma_{jk}\prod_{s\neq k}\lambda_s\right)$$

- We are, thus, very interested in the $\left(\gamma_{jk}\prod_{s\neq k}\lambda_s\right) =: \Gamma_{jk}$.

Particular approach

The last case explored will be if Γ_{jk} does not depend on k. If that is the case and $\mathcal{A}$ is a domain, then for all $l \neq j, k$,

$$\gamma_{jk} = \lambda_k \lambda_j^{-1}, \gamma_{jk}\lambda_k^{-1} = \gamma_{jl}\lambda_l^{-1}, \text{ but most importantly}, \Gamma_j = \prod_{s \neq j} \lambda_s.$$

This condition is very restrictive. Nonetheless, it still gives certain properties.

Proposition (GKS, Proposition 15)

Let $\mathcal{A}$ be a commutative associative algebra, $\lambda_i \in \mathcal{A}$, σ and τ two linear maps, $D_1, \ldots, D_n$ pairwise different (σ, τ)-derivations of $\mathcal{A}$ such that $D_i\sigma = \sigma D_i \cdot \lambda_i$ and $D_i\tau = \tau D_i \cdot \lambda_i$ for all i and $D_k D_j = D_j D_k \cdot \gamma_{jk}$, $\gamma_{jk} = \lambda_k \lambda_j^{-1}$ for all k. Each D_j is a generalized (σ, τ)-derivation with respect to the generalized Jacobian, with the following Leibniz-type rule:

$$[x_1, \ldots, x_n]_g D_j = \Gamma_j \cdot \sum_{i=1}^{n} [x_1\sigma, \ldots, x_{i-1}\sigma, x_i D_j, x_{i+1}\tau, \ldots, x_n\tau]_g.$$

State of the art

Theorem (GKS, Theorem 12)

Let Δ_s be the following sum of determinants:

$$\sum_{\substack{s=2\\i=1}}^{n} \begin{vmatrix} x_1\sigma D_1 & \dots & x_1\sigma D_n \\ \vdots & & \vdots \\ [x_i\sigma, y_2\sigma, y_{s-1}\sigma, \dots, y_s D_1, y_{s+1}\tau, \dots, y_n\tau]_g & \dots & [x_i\sigma, y_2\sigma, y_{s-1}\sigma, \dots, y_s D_n, y_{s+1}\tau \dots, y_n\tau]_g \\ \vdots & & \vdots \\ x_n\tau D_1 & \dots & x_n\tau D_n \end{vmatrix} \Gamma_i.$$

If $\Delta_s = 0$, then $(\mathcal{A}, [\star, \dots, \star]_g, \sigma, \tau)$ is a (σ, τ, n)-Hom-Lie algebra with Jacobi-type identity given by

$$[[x_1, \dots, x_n]_g, y_2\tau, \dots, y_n\tau]_g = \sum_{i=1}^{n} [x_1\sigma, \dots, x_{i-1}\sigma, [x_i, y_2, \dots, y_n]_g, x_{i+1}\tau, \dots, x_n\tau]_g.$$

Thank you

Thank you!