Primeness of groupoid graded rings

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Reference

This talk is based on the following paper:

Paula S. E. Moreira and Johan Öinert,
Prime groupoid graded rings with applications to partial skew groupoid rings
To appear in Communications in Algebra.

https://doi.org/10.1080/00927872.2024.2315311

Outline

Before we begin ...

In this talk, all rings are assumed to be associative, but not necessarily unital.

Reminder

A ring S is said to be prime if there are no nonzero ideals I, J of S such that $IJ = \{0\}$.

Related primeness investigations from the past

- Group rings:

Theorem (Connell)

$R[G]$ is prime iff R is prime and G has no non-triv. finite normal subgroup.

I. G. Connell (1963), On the group ring, Canad. J. Math. 15, 650–685.

- Unital strongly group graded rings:

D. S. Passman (1984), Infinite crossed products and group-graded rings, Trans. Amer. Math. Soc. 284(2), 707–727.

- (Non-unital) nearly epsilon-strongly group graded rings:

D. Lännström, P. Lundström, J. Öinert, S. Wagner (2021), Prime group graded rings with applications to partial crossed products and Leavitt path algebras, arXiv:2105.09224.

Groupoids: Conventions and notation

Throughout the rest of this talk, G denotes an arbitrary *groupoid*, i.e. a small category in which every morphism is invertible.

Notation:

- The set $\mathrm{mor}(G)$ will simply be denoted by G.
- Objects of G will be identified with their corresponding identity morphisms, allowing us to view the set $G_0 := \mathrm{ob}(G)$, as a subset of G.
- The *range* and *source* maps $r, s : G \to G_0$, indicate the range (codomain) respectively source (domain) of each morphism of G.
- The set of *composable pairs* of G is denoted by

$$G^2 := \{(g,h) \in G \times G : s(g) = r(h)\}.$$

- For each $e \in G_0$, we denote the corresponding *isotropy group* by

$$G_e^e := \{g \in G : s(g) = r(g) = e\}.$$

Groupoid graded rings

Definition

A ring S is said to be *G-graded* (or *graded by G*) if there is a collection $\{S_g\}_{g \in G}$ of additive subgroups of S such that

- $S = \oplus_{g \in G} S_g$,
- $S_g S_h \subseteq S_{gh}$, if $(g,h) \in G^2$, and
- $S_g S_h = \{0\}$, if $(g,h) \notin G^2$.

Examples

- Group graded rings. (Every group is a groupoid!)
- Partial skew groupoid rings
- Groupoid rings
- Leavitt path algebras graded by the "free path groupoid"
- Matrix rings. (Graded by pair groupoids.)

Before we continue: s-unitality

Reminder

A ring R is said to be s-unital if $a \in aR \cap Ra$ for every $a \in R$.

Example (A ring that is s-unital)

The ring of infinite matrices with complex entries of which only finitely many are nonzero.

Example (A ring that is NOT s-unital)

The ring $2\mathbb{Z}$.

Nearly epsilon-strong groupoid gradings

Definition (Lännström & Öinert)

Let S be a G-graded ring. We say that S is *nearly epsilon-strongly G-graded* if, for each $g \in G$ we have that

- $S_g S_{g^{-1}} S_g = S_g$, and
- $S_g S_{g^{-1}}$ is an s-unital ring.

Proposition

Let S be a G-graded ring. The following statements are equivalent:

1. *S is nearly epsilon-strongly G-graded;*
2. *For all $g \in G$ and $d \in S_g$, there exist $\epsilon_g(d) \in S_g S_{g^{-1}}$ and $\epsilon'_g(d) \in S_{g^{-1}} S_g$ such that $\epsilon_g(d) d = d \epsilon'_g(d) = d$.*

Basic properties of nearly epsilon-strongly graded rings

Proposition

Let S be a nearly epsilon-strongly G-graded ring. The following assertions hold:

1. S_e *is an* s*-unital ring, for every* $e \in G_0$.
2. $d \in d(\oplus_{e\in G_0} S_e) \cap (\oplus_{e\in G_0} S_e)d$, *for every* $d \in S$.
3. S *is* s*-unital and* $\oplus_{e\in G_0} S_e$ *is an* s*-unital subring of* S.
4. *Suppose that* H *is a subgroupoid of* G*. Then* $\oplus_{h\in H} S_h$ *is a nearly epsilon-strongly* H*-graded ring.*
5. $\oplus_{g\in G_e^e} S_g$ *is an* s*-unital ring, for every* $e \in G_0$.
6. *The set*

$$G' := \{g \in G : S_{s(g)} \neq \{0\} \text{ and } S_{r(g)} \neq \{0\}\}$$

is a subgroupoid of G.

7. $S = \oplus_{g\in G'} S_g$.

Invariance & graded ideals

Let S be a G-graded ring.

Definition

1. For any $g \in G$ and any subset I of S, we write $I^g := S_{g^{-1}} I S_g$.
2. Let H be a subgroupoid of G and let I be a subset of S. Then, I is called *H-invariant* if $I^g \subseteq I$ for every $g \in H$.

Definition

An ideal I of S is said to be *graded* (or *G-graded*) if $I = \oplus_{g \in G}(I \cap S_g)$.

A correspondence

The following maps are well defined:

$$\phi : \{G\text{-graded ideals of } S\} \ni I \mapsto I \cap \oplus_{e \in G_0} S_e \in \{G\text{-invariant ideals of } \oplus_{e \in G_0} S_e\}$$

$$\psi : \{G\text{-invariant ideals of } \oplus_{e \in G_0} S_e\} \ni J \mapsto SJS \in \{G\text{-graded ideals of } S\}$$

Theorem

Let S be a nearly epsilon-strongly G-graded ring. The map ϕ defines a bijection between the set of G-graded ideals of S and the set of G-invariant ideals of $\oplus_{e \in G_0} S_e$. The inverse of ϕ is given by ψ.

Graded primeness of groupoid graded rings

Definition

Let S be a G-graded ring.

1. $\oplus_{e \in G_0} S_e$ is said to be *G-prime* if there are no nonzero G-invariant ideals I, J of $\oplus_{e \in G_0} S_e$ such that $IJ = \{0\}$.
2. S is said to be *graded prime* if there are no nonzero G-graded ideals I, J of S such that $IJ = \{0\}$.

Theorem

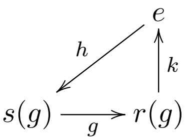
Let S be a nearly epsilon-strongly G-graded ring. Then S is graded prime if, and only if, $\oplus_{e \in G_0} S_e$ is G-prime.

Definition

Let S be a G-graded ring. An element $e \in G_0'$ is said to be a *support-hub* if for every nonzero $a_g \in S_g$, with $g \in G$, there are $h, k \in G$ such that $s(h) = e$, $r(k) = e$, and $a_g S_h$ and $S_k a_g$ are both nonzero.

Remark

- (a) Suppose that $e \in G_0'$ is a support-hub and that $a_g \in S_g$ is nonzero, for some $g \in G$. Notice that there are $h, k \in G$ as in the following diagram.



- (b) Notice that, if S is a ring which is nearly epsilon-strongly graded by a group G, then the identity element e of G is always a support-hub.

Support-hubs in relation to primeness

Proposition

Let S be a G-graded ring which is s-unital. If S is graded prime, then every $e \in G_0'$ is a support-hub.

Recall: G is *connected*, if $\forall e, f \in G_0$, $\exists g \in G$ such that $s(g) = e$ and $r(g) = f$.

Proposition

Let S be a G-graded ring which is s-unital. The following assertions hold:

1. *If G is a connected groupoid, then G' is a connected subgroupoid of G.*
2. *If there is a support-hub in G_0', then G' is a connected subgroupoid of G.*
3. *If S is graded prime, then G' is a connected subgroupoid of G.*

A couple of key results

Proposition

Let S be a nearly epsilon-strongly G-graded ring. If S is prime, then $\oplus_{g\in G_e^e} S_g$ is prime for every $e \in G_0$.

Theorem

Let S be a nearly epsilon-strongly G-graded ring. If there is some $e \in G_0'$ such that e is a support-hub and $\oplus_{g\in G_e^e} S_g$ is prime, then S is prime.

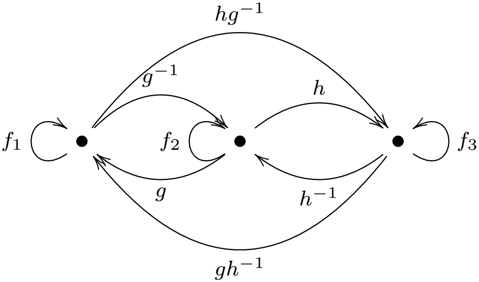
Remark

The assumption on the existence of a support-hub in the above theorem cannot be dropped. Indeed, consider the groupoid $G = \{e, f\} = G_0$ and the groupoid ring $S := \mathbb{C}[G]$. Then $G_e^e = \{e\}$ and $G_f^f = \{f\}$. Furthermore, $S_e \cong \mathbb{C}$ and $S_f \cong \mathbb{C}$ are both prime. Nevertheless, S is not prime.

Example: A support-hub in a connected grading groupoid is not enough to guarantee primeness

Example

Let $G := \{f_1, f_2, f_3, g, h, g^{-1}, h^{-1}, hg^{-1}, gh^{-1}\}$ be a groupoid with $G_0 = \{f_1, f_2, f_3\}$ and depicted as follows:



Example (Continued ...)

Define S as the ring of matrices over $\mathbb{Z}$ of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

Denote by $\{e_{ij}\}_{i,j}$, the canonical basis of S and define:

$$S_g := \mathbb{Z}e_{12}, \qquad S_{g^{-1}} := \mathbb{Z}e_{21}, \qquad S_h := \mathbb{Z}e_{43}, \qquad S_{h^{-1}} := \mathbb{Z}e_{34},$$

$$S_{f_1} := \mathbb{Z}e_{11}, \qquad S_{f_3} := \mathbb{Z}e_{44}, \qquad S_{f_2} := \{\lambda_1 e_{22} + \lambda_2 e_{33} : \lambda_1, \lambda_2 \in \mathbb{Z}\},$$

and $S_l := \{0\}$, otherwise. This G-grading is nearly epsilon-strong! One can show that $S = \oplus_{l \in G} S_l$ and that $f_2 \in G_0'$ is a support-hub. Note that $e_{12} \in S_g$ and $e_{43} \in S_h$ are nonzero elements, but there is no element $a_l \in S_l$ such that $l \in G'$ and $e_{12} a_l e_{43} \neq 0$. It follows that S is not graded prime.

Our main result

Theorem

Let S be a nearly epsilon-strongly G-graded ring, and let $G' := \{g \in G : S_{s(g)} \neq \{0\} \text{ and } S_{r(g)} \neq \{0\}\}$. The following statements are equivalent:

1. (i) *S is prime;*
2. (ii) *$\bigoplus_{e \in G_0} S_e$ is G-prime, and for every $e \in G'_0$, $\bigoplus_{g \in G_e^e} S_g$ is prime;*
3. (iii) *$\bigoplus_{e \in G_0} S_e$ is G-prime, and for some $e \in G'_0$, $\bigoplus_{g \in G_e^e} S_g$ is prime;*
4. (iv) *S is graded prime, and for every $e \in G'_0$, $\bigoplus_{g \in G_e^e} S_g$ is prime;*
5. (v) *S is graded prime, and for some $e \in G'_0$, $\bigoplus_{g \in G_e^e} S_g$ is prime;*
6. (vi) *For every $e \in G'_0$, e is a support-hub, and $\bigoplus_{g \in G_e^e} S_g$ is prime;*
7. (vii) *For some $e \in G'_0$, e is a support-hub, and $\bigoplus_{g \in G_e^e} S_g$ is prime.*

Groupoid rings

Ingredients: An s-unital ring R, and a groupoid G.

The corresponding *groupoid ring* $R[G]$ consists of elements of the form

$$\sum_{g \in G} a_g \delta_g$$

where $a_g \in R$ is zero for all but finitely many $g \in G$.

- Addition: The natural one
- Multiplication: For $g, h \in G$ and $a, b \in R$, the multiplication in $R[G]$ is defined by the rule

$$a\delta_g \cdot b\delta_h := \begin{cases} ab\delta_{gh} & \text{if } (g,h) \in G^2, \\ 0 & \text{otherwise.} \end{cases}$$

A characterization of prime groupoid rings

Theorem

Let G be a groupoid and let R be a nonzero s-unital ring. The following assertions are equivalent:

1. *The groupoid ring $R[G]$ is prime;*
2. *G is connected and there is some $e \in G_0$ such that the group ring $R[G_e^e]$ is prime;*
3. *G is connected and, for every $e \in G_0$, the group ring $R[G_e^e]$ is prime;*
4. *G is connected, R is prime and there is some $e \in G_0$ such that G_e^e has no non-trivial finite normal subgroup;*
5. *G is connected, R is prime and, for every $e \in G_0$, G_e^e has no non-trivial finite normal subgroup.*

The end

THANK YOU FOR YOUR ATTENTION!