A hom-associative Stafford's theorem?

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Stafford's theorem says that every one-sided ideal in the Weyl algebras A_n over a field of characteristic zero is generated by two elements.

Strengthening of the fact that the A_n are Noetherian. We wanted a hom-associative version of Stafford's theorem.

Hom-associative rings

A hom-associative ring (R, α) is a non-associative ring R together with an additive function $\alpha : R \to R$ such that

$$\alpha(a)(bc) = (ab)\alpha(c)$$

for all $a, b, c \in R$.
Introduced by Makhlouf and Silvestrov. Connection with Hom-Lie algebras which were defined earlier by Hartwig, Larsson and Silvestrov.
Include the non-associative rings as a special case with $\alpha \equiv 0$.

Yau twist

Theorem ([3, 4])

Let A be an associative algebra and let α be an algebra endomorphism on A. Define a new product $*$ on A by $a * b := \alpha(ab)$ for any $a, b \in A$. Then A with product $*$, called the Yau twist of A, is a hom-associative algebra with twisting map α.

The old identity element, 1_A, now satisfies $1_A * a = a * 1_A = \alpha(a)$.

Weyl algebras

The *nth Weyl algebra*, A_n, over a field K of characteristic zero is the free, associative, and unital algebra with generators $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$, $K\langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \rangle$, modulo the commutation relations

$$x_i x_j = x_j x_i \text{ for all } i, j \in \{1, 2, \ldots, n\},$$
$$y_i y_j = y_j y_i \text{ for all } i, j \in \{1, 2, \ldots, n\},$$
$$x_i y_j = y_j x_i \text{ for all } i, j \in \{1, 2, \ldots, n\} \text{ such that } i \neq j,$$
$$x_i y_i = y_i x_i + 1 \text{ for all } i \in \{1, 2, \ldots, n\}.$$

First Hom-associative Weyl algebra

In [1], a family of *hom-associative Weyl algebras* $\{A_1^k\}_{k\in K}$ was constructed as a generalization of A_1 to the hom-associative setting. The definition of A_1^k is as follows:
Let α_k be the K-automorphism on A_1 defined by $\alpha_k(x) := x$, $\alpha_k(y) := y + k$, and $\alpha_k(1_{A_1}) := 1_{A_1}$ for any $k \in K$. The *first hom-associative Weyl algebra* A_1^k is the Yau twist of A_1 by α_k.
In the same article it is proven that A_1^k is simple.

In [2], the study of A_1^k was continued. The morphisms and derivations on A_1^k were characterized, and an analogue of the famous *Dixmier conjecture*, first introduced by Dixmer, was proven. It was also shown that A_1^k is a formal deformation of A_1 with k as deformation parameter, this in contrast to the associative setting where A_1 is formally rigid and thus cannot be formally deformed.

Higher hom-associaive Weyl algebras

Let K be a field of characteristic zero and let $k = (k_1, k_2, \ldots, k_n) \in K^n$. Define the K-automorphism α_k on A_n by $\alpha_k(x_i) := x_i$, $\alpha_k(y_i) := y_i + k_i$, and $\alpha_k(1_{A_n}) := 1_{A_n}$ for $1 \leq i \leq n$. The *nth hom-associative Weyl algebra* A_n^k is the Yau twist of A_n by α_k.

Proposition

If I is a left (right) ideal of A_n, then I is a left (right) ideal of A_n^k if and only if $\alpha_k(I) \subseteq I$.

Proposition

Any left (right) ideal of A_n^k for $k_1 k_2 \cdots k_n \neq 0$ is a left (right) ideal of A_n.

Proof that $\alpha_k(I) = I$

let I be a left ideal of A_n^k. If $0 \neq p \in I$, we claim that we can find an element $p' \in I$ such that $\deg_x(p') = \deg_x(p)$ and $L(p') = 1$. Now suppose $I \not\subseteq \alpha_k(I)$. Then there is at least one element in I that does not belong to $\alpha_k(I)$. Pick such an element, q, of lowest possible degree w.r.t. $x_1, x_2, \ldots, x_n$. Find an element $q' \in I$ such that $\deg_x(q') = \deg_x(q)$ and $L(q') = 1$. Set $r = \alpha_k^2(\alpha_k^{-2}(L(q))q') = \alpha_k(\alpha_k^{-2}(L(q)) * q')$. Note that $\deg_x(r) = \deg_x(q)$ and that $L(r) = L(q)$. Since $\alpha_k^{-2}(L(q)) * q' \in I$, we have $r \in \alpha_k(I) \subseteq I$. Hence $q - r \in I$, and by the minimality of q, we must have $q - r \in \alpha_k(I)$. However, this would imply that also $q \in \alpha_k(I)$, which is a contradiction.

Hom-associative Stafford

Corollary

Any left (right) ideal of A_n^k is generated by two elements.

Proof.

Let I be a left ideal of A_n^k. Since I is also a left ideal of A_n, we know that it is generated as an ideal of A_n by two elements, say p and q. We want to show that $\alpha^{-1}(p)$ and $\alpha^{-1}(q)$ generate I as a left ideal of A_n^k. If $r \in I$, then there are $a, b \in A_n$ such that $r = ap + bq = \alpha(\alpha^{-1}(a)\alpha^{-1}(p) + \alpha^{-1}(b)\alpha^{-1}(q)) = \alpha^{-1}(a) * \alpha^{-1}(p) + \alpha^{-1}(b) * \alpha^{-1}(q)$. Clearly this shows that $\alpha^{-1}(p)$ and $\alpha^{-1}(q)$ generate I as a left ideal of A_n^k.

The right case is similar. □

So we have proven a hom-associative Stafford's theorem. But it turns out one can go further.

Lemma

For any $p \in A_n$, there are $q_1, q_2, \ldots, q_m \in A_n$ with $\deg_y(q_1) = \deg_y(q_2) = \cdots = \deg_y(q_m) = 0$, such that the left (right) ideal of A_n^k for $k_1 k_2 \cdots k_n \neq 0$ generated by p equals the left (right) ideal of A_n^k generated by $q_1, q_2, \ldots, q_m$.

Theorem

Any left (right) ideal of A_n^k for $k_1 k_2 \cdots k_n \neq 0$ is principal.

Proof.

Let I be a left ideal of A_n^k. We know it is generated by elements p, q as a left ideal of A_n. By 2, we can find $p_1, p_2, \dots, p_\ell$ and $q_1, q_2, \dots, q_m$ such that the p_i generate the same left ideal of A_n^k as p, the q_i generate the same left ideal of A_n^k as q, and $\deg_y(p_1) = \deg_y(p_2) = \cdots = \deg_y(p_\ell) = \deg_y(q_1) = \deg_y(q_2) = \cdots = \deg_y(q_m) = 0$. Clearly $p_1, p_2, \dots, p_\ell, q_1, q_2, \dots, q_m$ generate I as a left ideal of A_n^k. By **??** we are done.

The right case is similar. □

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