Chain algebras of finite distributive lattices

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Toric rings / Monomial subalgebras

$\mathcal{M}$ = a finite set of monomials in a polynomial ring $k[x_1, \dots, x_n]$.

$k[\mathcal{M}]$ = subring generated by $\mathcal{M}$

$$k[\mathcal{M}] \cong R/I_{\mathcal{M}}$$

$\mathcal{M} = \{m_1, \dots, m_s\}$ $\quad R = k[y_1, \dots, y_s]$

$\varphi : R \to k[\mathcal{M}]$ $\quad y_i \mapsto m_i$

$I_{\mathcal{M}} = \ker \varphi$ $\quad$ a binomial prime ideal $\quad$ (toric ideal)

If the monomials in $\mathcal{M}$ have the same degree, then $I_{\mathcal{M}}$ is homogeneous.

Example: Twisted cubic

$$k[x^3, x^2y, xy^2, y^3] \cong \frac{k[z_1, z_2, z_3, z_4]}{(z_1z_4 - z_2z_3,\ z_1z_3 - z_2^2,\ z_2z_4 - z_3^2)}$$

Toric rings in combinatorial commutative algebra

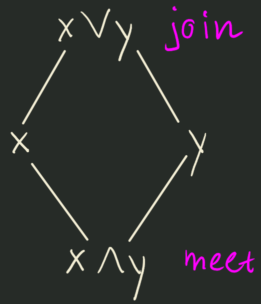
$\mathcal{M}$ = monomials given by a combinatorial object.

- Edge ring of a graph
- Base ring of a matroid
- Hibi ring defined by a poset
- $\vdots$

Properties of the underlying combinatorial object $\rightsquigarrow$ algebraic properties of $\Bbbk[\mathcal{M}]$

Posets & lattices

Poset = (finite) set with a partial order $<$.



If any pair x, y has a meet and a join then the poset is called a <u>lattice</u>.

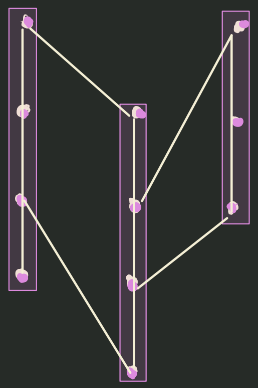
<u>Order ideal</u> of a poset = downwards closed set

$J(P)$ = set of order ideals of P $\rightsquigarrow$ new poset

distributive lattice

Thm (Birkhoff): Any finite distributive lattice is identical to $J(P)$ for some poset P.

$L = J(P)$ P can be obtained from L by taking the join irreducibles.

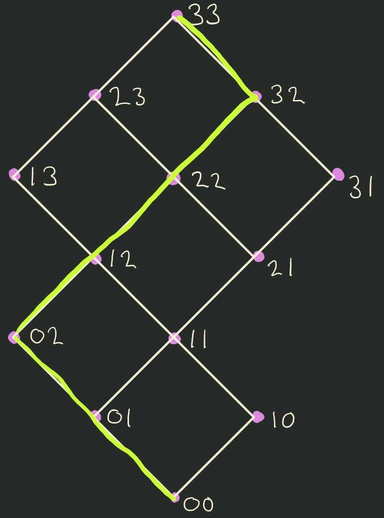


width(P) = #chains needed to cover P

$L = J(P)$ $\dim(L) = \text{width}(P)$

Chain algebra of a finite distributive lattice L

maximal chain in L $\longleftrightarrow$ monomial in $k[x_{00}, x_{01}, \ldots, x_{33}]$



$x_{00}\, x_{01}\, x_{02}\, x_{12}\, x_{22}\, x_{32}\, x_{33}$

$\mathcal{C}_L = \{\text{all monomials from maximal chains of } L\}$

$$k[\mathcal{C}_L] \cong R/I_{\mathcal{C}_L}$$

maximal chains in a finite distributive lattice all have the same length $\rightsquigarrow$ $I_{\mathcal{C}_L}$ is homogeneous ideal

Some general results. L finite distributive lattice

- $I_{\mathcal{C}_L}$ has a squarefree initial ideal w.r.t. any DegRevLex term order.

Sturmfels, Hochster

- $k[\mathcal{C}_L]$ is normal and Cohen-Macaulay

- Krull dimension of $k[\mathcal{C}_L]$ is $|L| - |P|$, where $L = J(P)$

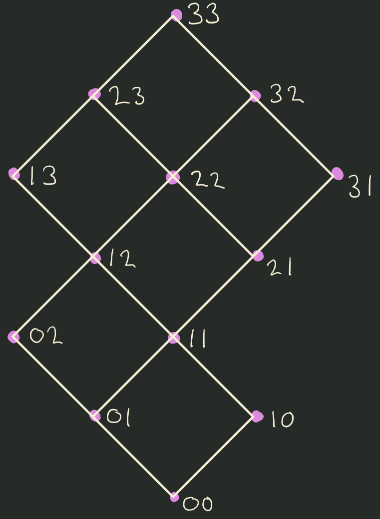
$\dim k[\mathcal{M}] = \mathrm{rank}\begin{bmatrix}\text{exponent vectors} \\ \text{of monomials } \mathcal{M}\end{bmatrix}$

$\text{col. space}\begin{bmatrix}\text{exponent vectors} \\ \text{of monomials in } \mathcal{C}_L\end{bmatrix} = \mathrm{span}\left\{\begin{matrix}\text{exp. vec. of} \\ \text{one monomial}\end{matrix},\ \text{columns}\begin{bmatrix}\text{incidence} \\ \text{matrix of} \\ \text{a directed graph}\end{bmatrix}\right\}$

linear algebra + graph theory

Planar f. d. lattices L

$\dim(L) \leq 2$ Can be embedded in $\mathbb{N}^2$



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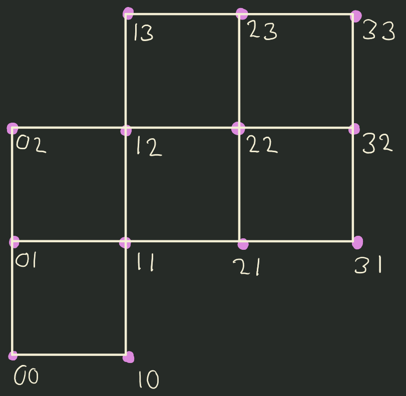

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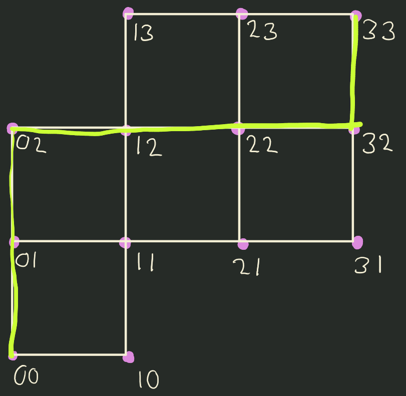
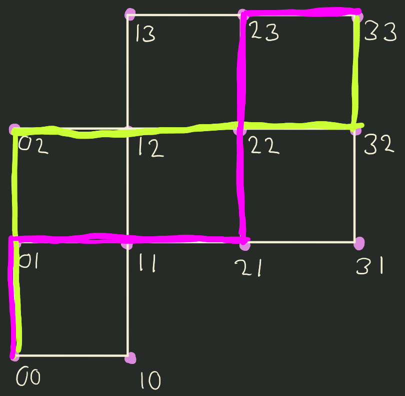
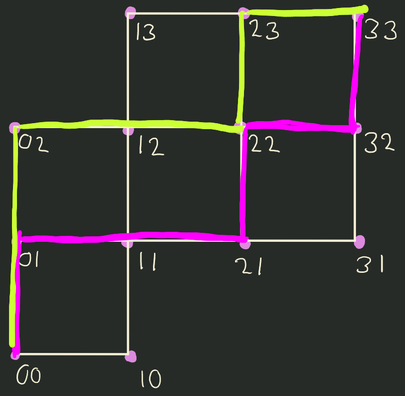


maximal chain in L $\rightsquigarrow$
north-east path from 00 to 33.





$$(x_{00}x_{01}x_{02}x_{12}x_{22}x_{32}x_{33})\cdot(x_{00}x_{01}x_{11}x_{21}x_{22}x_{23}x_{33}) = (x_{00}x_{01}x_{02}x_{12}x_{22}x_{23}x_{33})\cdot(x_{00}x_{01}x_{11}x_{21}x_{22}x_{32}x_{33})$$

$y_1 y_2 - y_3 y_4$ in I_{C_L}

I_{C_L} is generated by degree 2 binomials of this type.

__Theorem.__ L is planar $\Longleftrightarrow$ $I_{\mathcal{C}_L}$ is quadratic

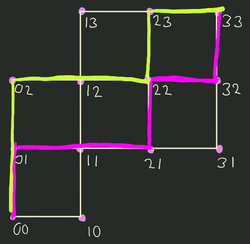
Moreover, when L is planar $I_{\mathcal{C}_L}$ has a quadratic Gröbner basis.

$k[\mathcal{C}_L]$ is a Koszul algebra

Hilbert series of $k[\mathcal{C}_L] = \bigoplus_{i \geq 0} A_i$ $\quad$ $A_i = \operatorname{span}\{\text{products of } i \text{ monomials from } \mathcal{C}_L\}$

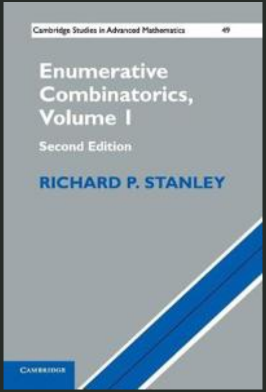
Hilbert function of $k[\mathcal{C}_L]$: $\quad HF(i) = \dim_k A_i$

$= \#\, i$-tuples of non-crossing north-east paths

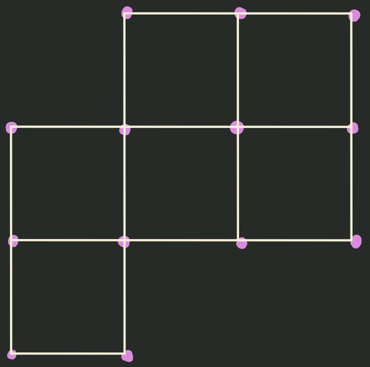


Hilbert series: $\sum_{i \geq 0} HF(i) z^i$

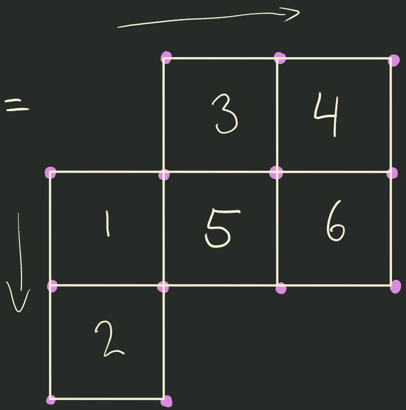
$$= \sum_{\substack{\text{SYT's } T \\ \text{shape } L}} z^{\operatorname{asc}(T)} \Big/ (1-z)^d$$



Shape given by L

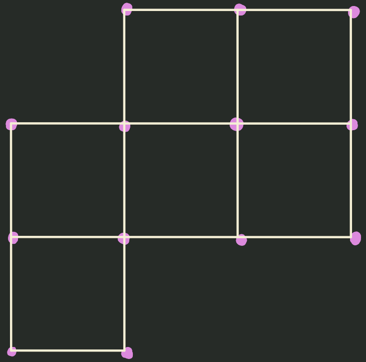


$T =$

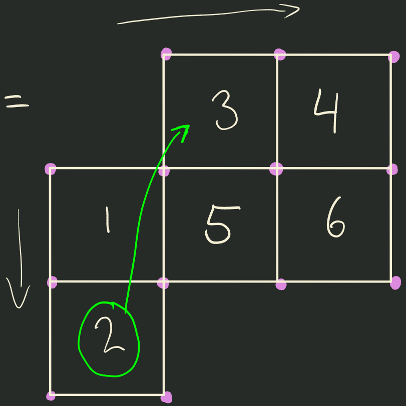


Standard Young tableaux

Shape given by λ



$T =$



Standard Young tableaux

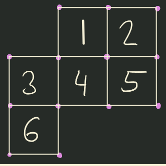
ascent : i s.t $i+1$ sits above i in T.

$asc(T) = \#$ ascents in T

ascents | SYT's

$$\frac{1+8z+10z^2+2z^3}{(1-z)^7}$$

0



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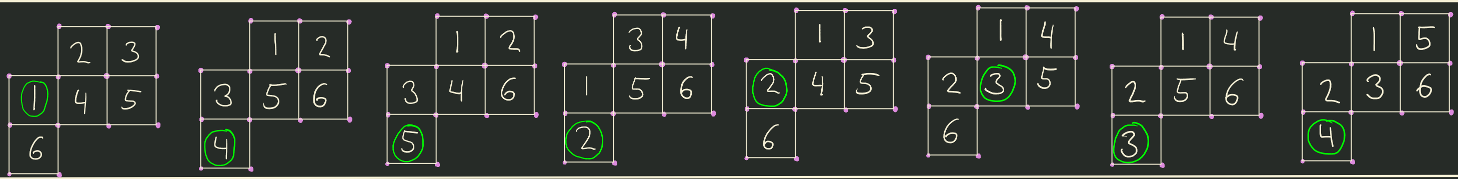
1



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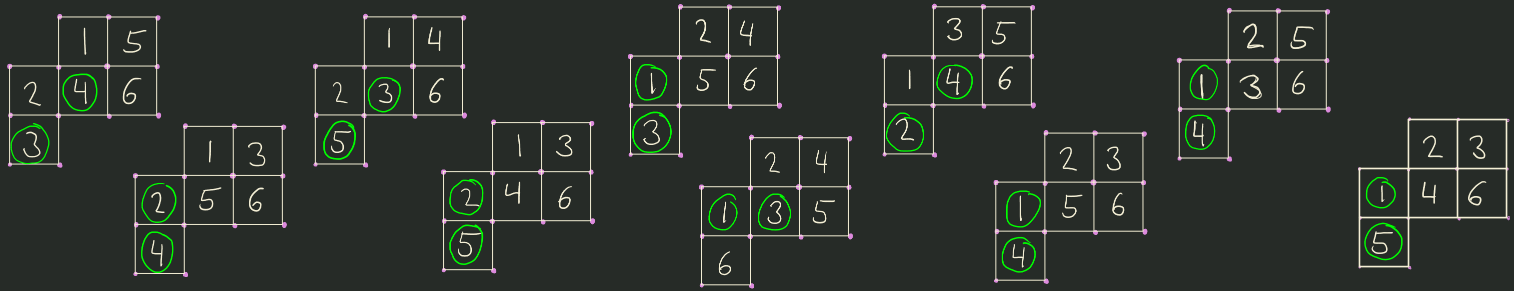
2



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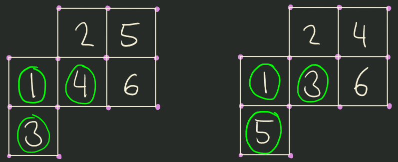
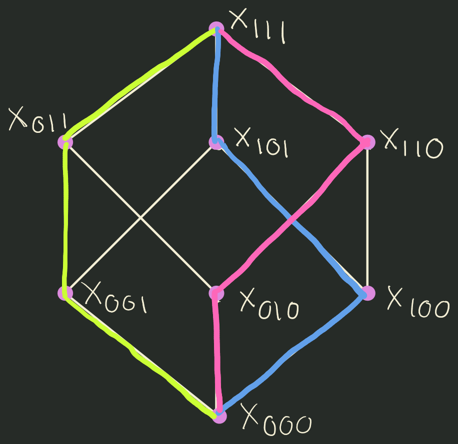
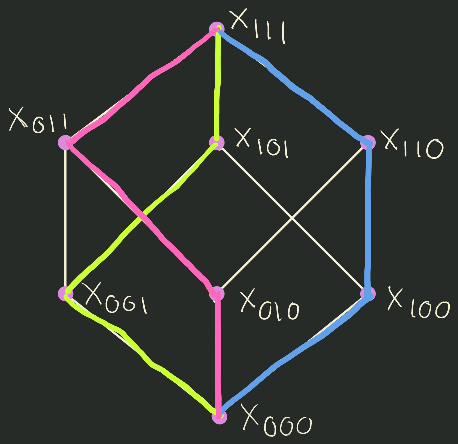
3



2

The non-planar case





$$x_{000}x_{001}x_{011}x_{111}\cdot x_{000}x_{010}x_{110}x_{111}\cdot x_{000}x_{100}x_{101}x_{111} = x_{000}x_{001}x_{101}x_{111}\cdot x_{000}x_{010}x_{011}x_{111}\cdot x_{000}x_{100}x_{110}x_{111}$$

$$k[C_h] \cong \frac{k[y_1,\dots,y_6]}{(y_1y_2y_3 - y_4y_5y_6)}$$

Theorem: $I_{\mathcal{C}_h}$ has a minimal generator of degree $\dim(h)$.

$I_{\mathcal{C}_h}$ is generated by binomials of degrees $\leq \frac{|\mathcal{C}_h|}{2}$.

Open problem: Give a better description of $I_{\mathcal{C}_h}$ in the non-planar case!

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