A new perspective on the Cayley–Dickson construction: flipped polynomial rings

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Outline

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Background and motivation

This talk is based on joint work with M. Aryapoor (MDU); [AB24].

Conventions. All rings in this talk are unital, but not necessarily commutative.
A *non-associative ring* is a ring which is not necessarily associative.

[AB24] M. Aryapoor and P. Bäck. "Flipped non-associative polynomial rings and the Cayley–Dickson construction". In: arXiv:2403.03763 (2024).

The Cayley–Dickson construction, attributed to A. Cayley [Cay45] and L. E. Dickson [Dic19], generates new $*$-algebras out of old ones; e.g. $\mathbb{C}, \mathbb{H}, \mathbb{O}, \ldots$

However, the construction is mysterious! That of $\mathbb{C} \cong \mathbb{R}[X]/(X^2+1)$ and $\mathbb{H} \cong \mathbb{C}[X;*]/(X^2+1)$ is not. Is there a class of polynomial rings lurking behind it *all*?

Non-commutative rings with a *skewed* or *twisted* multiplication; *Hilbert's twist* [Hil03]. *Ore extensions* were later introduced by Ø. Ore [Ore33].

Appear as *universal enveloping algebras* of Lie algebras, *quantized coordinate rings* of affine algebraic varieties, *differential operator rings* etc. Used e.g. in coding theory.

Non-associative Ore extensions introduced in [NÖR18], are part of the answer to it *all*!

[Cay45] A. Cayley. "On Jacobi's elliptic functions, in reply to Rev. B. Bronwin; and on quaternions". In: *Philos. Mag.* 26.3 (1845).

[Dic19] L. E. Dickson. "On quaternions and their generalization and the history of the eight square theorem". In: *Ann. Math* 20.3 (1919).

[Hil03] D. Hilbert. *Grundlagen der Geometrie*. Leipzig: Teubner, 1903.

[Ore33] O. Ore. "Theory of non-commutative polynomials". In: *Ann. Math* 34.3 (1933).

[NÖR18] P. Nystedt, J. Öinert, and J. Richter. "Non-associative Ore extensions". In: *Isr. J. Math.* 224.1 (2018).

Non-commutative polynomial rings: Ore extensions

Definition (Ore extension)

Let S be a non-associative ring, $R \subseteq S$ with $1 \in R$. S is an *Ore extension of* R if these axioms hold:

(O1) There is an $x \in S$ s.t. S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(O2) $xR \subseteq Rx + R$;

(O3) S is associative.

Let R be an associative ring with an endomorphism σ and a *left σ-derivation* δ:

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s, \quad \text{for any } r, s \in R.$$

The *ordinary generalized polynomial ring* $R[X; \sigma, \delta]$ is $\left\{ \sum_{i \in \mathbb{N}} r_i X^i : r_i \in R \text{ zero for all but finitely many } i \in \mathbb{N} \right\}$ with pointwise addition and

$$(rX^m)(sX^n) = \sum_{i \in \mathbb{N}} \left(r\pi_i^m(s) \right) X^{i+n}, \quad \text{for any } r, s \in R, m, n \in \mathbb{N},$$

$\pi_i^m : R \to R$ the sum of all $\binom{m}{i}$ composition of i copies of σ and $m - i$ copies of δ.

Proposition

$R[X; \sigma, \delta]$ is an Ore extension of R with $x = X$; every Ore extension of R is isomorphic to an ordinary generalized polynomial ring $R[X; \sigma, \delta]$.

Theorem (Hilbert's basis theorem)

Let σ be an automorphism on R. If R is right (left) Noetherian, then so is $R[X;\sigma,\delta]$.

Example

Let R be an associative ring. Then $R[X] = R[X;\mathrm{id}_R,0]$.

If R is an associative ring, $\sigma = \mathrm{id}_R$, we get a *differential polynomial ring* $R[X;\mathrm{id}_R,\delta]$.

Example

Let K be a field. The *first Weyl algebra* is $K\langle X,Y\rangle/(XY-YX-1) \cong K[Y][X;\mathrm{id}_{K[Y]},\mathrm{d}/\mathrm{d}Y]$.
In quantum physics, X and Y are position and momentum operators.

If $\delta = 0$, we get a *skew polynomial ring* $R[X;\sigma,0]$, denoted by $R[X;\sigma]$.

Example

Let $*\colon \mathbb{C} \to \mathbb{C}$, $u \mapsto u^*$ be complex conjugation. In $\mathbb{C}[X;*]$, $Xu = u^*X$.
We have $\mathbb{C} \cong \mathbb{R}[X]/(X^2+1)$ and $\mathbb{H} \cong \mathbb{C}[X;*]/(X^2+1)$.

Non-associative Ore extensions

Definition (Associators, commutators etc.)

$[\cdot,\cdot]\colon R \times R \to R$ is defined by $[r,s] := rs - sr$,
$(\cdot,\cdot,\cdot)\colon R \times R \times R \to R$ is defined by $(r,s,t) := (rs)t - r(st)$ for any $r,s,t \in R$.
(A,B,C) finite sums (a,b,c) with $a \in A$, $b \in B$, $c \in C$ for $A,B,C \subseteq R$.

$N_l(R) := \{r \in R\colon (r,s,t) = 0 \text{ for all } s,t \in R\}$. $N_m(R)$ and $N_r(R)$ defined similarly.
$N(R) := N_l(R) \cap N_m(R) \cap N_r(R)$.

$C(R) := \{r \in R\colon [r,s] = 0 \text{ for all } s \in R\}$.
$Z(R) := C(R) \cap N(R)$.

Definition (Left R-module)

If R is a non-associative ring, a *left R-module* is an additive group M with a biadditive map $R \times M \to M$, $(r,m) \mapsto rm$ for any $r \in R$ and $m \in M$.

Let R be a **non-associative ring** with **additive maps σ, δ** where $\sigma(1) = 1$, $\delta(1) = 0$. The *generalized polynomial ring* $R[X; \sigma, \delta]$ is defined as in the associative case,

$$(rX^m)\,(sX^n) = \sum_{i \in \mathbb{N}} \left(r\pi_i^m(s)\right) X^{i+n}, \quad \text{for any } r, s \in R,\; m, n \in \mathbb{N}.$$

Definition (Non-associative Ore extension)

Let S be a non-associative ring, $R \subseteq S$ with $1 \in R$. S is a ***non-associative*** *Ore extension of R* if these axioms hold:

(N1) There is an $x \in S$ s.t. S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(N2) $xR \subseteq Rx + R$;

(N3) $(S, S, x) = (S, x, S) = \{0\}$.

Proposition ([NÖR18])

*$R[X; \sigma, \delta]$ is a **non-associative** Ore extension of R with $x = X$; every **non-associative** Ore extension of R is isomorphic to a generalized polynomial ring $R[X; \sigma, \delta]$.*

Theorem (Hilbert's basis theorem for non-associative Ore extensions [BR23])

Let σ be an additive bijection, $\sigma(1) = 1$. If R is right Noetherian, then so is $R[X; \sigma, \delta]$.

There is a left Noetherian ring R, a σ, and a δ s.t. $R[X; \sigma, \delta]$ is *not* left Noetherian.

[BR23] P. Bäck and J. Richter. "Hilbert's basis theorem and simplicity for non-associative skew Laurent polynomial rings and related rings". In: arXiv:2207.07994 (2023).

Flipped polynomial rings

Definition (Non-associative polynomial ring)

Let A be an additive group. A *non-associative polynomial ring over A* is the additive group $A[X]$ equipped with a non-associative ring structure.

{Non-associative polynomial rings over A} $\longleftrightarrow$
{Collections of biadditive maps $\ell_{m,n;k} : A \times A \to A$, $m, n, k \in \mathbb{N}$, only finitely many $k \in \mathbb{N}$ s.t. $\ell_{m,n;k}(a, b) \neq 0$},

$$(aX^m)(bX^n) = \sum_{k\in\mathbb{N}} \ell_{m,n;k}(a, b)X^k.$$

Let $\tau : A \times A \to A \times A$ be the *flip map*, i.e. $\tau(a, b) := (b, a)$ for any $a, b \in A$.

Definition (Flipped non-associative polynomial ring)

Let S be a non-associative polynomial ring corresponding to a collection of $\ell_{m,n;k}$. The *flipped* ditto S^{fl} is the one corresponding to the collection of $\ell_{m,n;k} \circ \tau^n$.

Note. Since $\ell_{m,n;k} \circ \tau^n \circ \tau^n = \ell_{m,n;k}$, $(S^{\mathrm{fl}})^{\mathrm{fl}} = S$, similarly to $(S^{\mathrm{op}})^{\mathrm{op}} = S$.

Flipped polynomial rings

Every generalized polynomial ring $R[X;\sigma,\delta]$ now has a flipped cousin, $R[X;\sigma,\delta]^{\mathsf{fl}}$:

$$(rX^m)(sX^n) = \sum_{i\in\mathbb{N}} \tau_n(r, \pi_i^m(s))X^{i+n},$$

where

$$\tau_n(r,s) := \begin{cases} rs & \text{if } n \text{ is even,} \\ sr & \text{if } n \text{ is odd.} \end{cases}$$

Example

In $R[X]^{\mathsf{fl}} = R[X;\mathsf{id}_R, 0]^{\mathsf{fl}}$, $(rX^m)(sX^n) = \tau_n(r,s)X^{m+n}$, so $r(sX) = (rX^0)(sX^1) = (sr)X$.

Definition (Flipped non-associative Ore extension)

Let S be a non-associative ring, $R \subseteq S$ with $1 \in R$. S is a ***flipped*** *non-associative Ore extension of* R if these axioms hold:

(F1) There is an $x \in S$ s.t. S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(F2) $xR \subseteq Rx + R$;

(F3) The following identities hold for any $m, n \in \mathbb{N}$ and $r, s \in R$:

$$(rx^{m+1})(sx^n) = ((rx^m)(\sigma(s)x^n))x + (rx^m)(\delta(s)x^n), \quad r(sx^n) = \tau_n(r,s)x^n.$$

σ and δ are additive maps s.t. $xr = \sigma(r)x + \delta(r)$ for all $r \in R$; such exist by (F1) and (F2).

Proposition ([AB24])

*$R[X;\sigma,\delta]^{\mathrm{fl}}$ is a **flipped** non-associative Ore extension of R with $x = X$; every **flipped** non-associative Ore extension of R is isomorphic to a **flipped** generalized polynomial ring $R[X;\sigma,\delta]^{\mathrm{fl}}$.*

Proposition ([AB24])

If $S = R[X;\sigma,\delta]^{\mathsf{fl}}$, then

(i) $X \in N_l(S)$ *iff* σ *is an endomorphism and* δ *is both a left and a right* σ*-derivation;*

(ii) $X \in N_m(S)$ *iff* $\mathbf{im}(\delta^n \circ \sigma) \subseteq C(R)$ *for any* $n \in \mathbb{N}$*;*

(iii) $X \in N_r(S)$ *iff* R *is commutative.*

Proposition ([AB24])

$R[X;\sigma,\delta]^{\mathsf{fl}}$ *is associative iff* R *is associative and commutative,* σ *is an endomorphism and* δ *is a left and a right* σ*-derivation.*

Theorem (Hilbert's basis theorem for flipped non-associative Ore extensions [AB24])

Let σ *be an additive bijection,* $\sigma(1) = 1$*. If* R *is right Noetherian, then so is* $R[X;\sigma,\delta]^{\mathsf{fl}}$*.*

There is a left Noetherian ring R, a σ, and a δ s.t. $R[X;\sigma,\delta]^{\mathsf{fl}}$ is *not* left Noetherian.

The Cayley–Dickson construction

A $*$-*algebra* A is a non-associative algebra over an associative, commutative ring K with an *involution* $*\colon A \to A$, $a \mapsto a^*$ ($(ab)^* = b^*a^*$, $(a^*)^* = a$ for any $a, b \in A$).

Let $\mu \in K$ where for all $a \in A$, $k \in K$, $\mu a = 0 \implies a = 0$ and $kA = 0 \implies k = 0$. The *Cayley double of* A, $\mathrm{Cay}(A, \mu)$, is $A \oplus A$ where for all $a, b, c, d \in A$,

$$(a, b)(c, d) := (ac + \mu d^* b, da + bc^*),$$
$$(a, b)^* := (a^*, -b).$$

Example

Start with $* = \mathrm{id}_K$ on $K = \mathbb{R}$, choose $\mu = \pm 1$ and then double:

$$\begin{aligned} \mathrm{Cay}(\mathbb{R}, -1) &\cong \mathbb{C}, & \mathrm{Cay}(\mathbb{R}, +1) &\cong \mathbb{C}' \\ \mathrm{Cay}(\mathbb{C}, -1) &\cong \mathbb{H}, & \mathrm{Cay}(\mathbb{C}', +1) &\cong \mathbb{H}' \\ \mathrm{Cay}(\mathbb{H}, -1) &\cong \mathbb{O}, & \mathrm{Cay}(\mathbb{H}', +1) &\cong \mathbb{O}', \\ &\vdots & &\vdots \end{aligned}$$

Any non-associative $*$-algebra A gives rise to a flipped skew polynomial ring $A[X;*]^{\mathrm{fl}}$,

$$(aX^m)(bX^n) = \tau_n(a, *^m(b))X^{m+n}, \quad \text{for any } a, b \in A,\ m, n \in \mathbb{N}.$$

We can make $A[X;*]^{\mathrm{fl}}$ a $*$-algebra by extending $*$,

$$(a_0 + a_1X + a_2X^2 + \cdots)^* := a_0^* - a_1X + a_2^*X^2 - \cdots, \quad \text{for any } a_0, a_1, a_2, \ldots \in A.$$

If $*$ is nontrivial, $2A \neq 0$, and A contains no zero divisors, there is only one more way:

$$(a_0 + a_1X + a_2X^2 + \cdots)^* := a_0^* + a_1X + a_2^*X^2 + \cdots, \quad \text{for any } a_0, a_1, a_2, \ldots \in A.$$

Theorem ([AB24])

$\mathrm{Cay}(A, \mu)$ *and* $A[X;*]^{\mathrm{fl}}/(X^2 - \mu)$ *are isomorphic as* $*$*-algebras.*

In particular, we can now see the (non-)dark side of the moon:

$$\mathbb{C} \cong \mathbb{R}[X]/(X^2+1), \qquad \mathbb{C}' \cong \mathbb{R}[X]/(X^2-1),$$
$$\mathbb{H} \cong \mathbb{C}[X;*]/(X^2+1), \qquad \mathbb{H}' \cong \mathbb{C}'[X;*]/(X^2-1),$$
$$\mathbb{O} \cong \mathbb{H}[X;*]^{\mathrm{fl}}/(X^2+1), \qquad \mathbb{O}' \cong \mathbb{H}'[X;*]^{\mathrm{fl}}/(X^2-1),$$
$$\vdots \qquad\qquad \vdots$$

We can view $A[t]$ as a non-associative $*$-algebra over $K[t]$ by letting $t^* := t$.

Theorem ([AB24])

$\mathrm{Cay}(A[t], t)$ *and* $A[X; *]^{\mathrm{fl}}$ *are isomorphic as* $*$*-algebras.*

A is *flexible* if $(a, b, a) = 0$, and *alternative* if $(a, a, b) = (b, a, a) = 0$ for all $a, b \in A$.

We get counterparts to classical results on Cayley–Dickson algebras by K. McCrimmon [McC85]:

Theorem ([AB24])

If $B = A[X; *]^{\mathrm{fl}}$*, then the following assertions hold:*

(i) *The involution on* B *is trivial iff the involution* $*$ *on* A *is trivial and* $2A = 0$*;*

(ii) B *is commutative iff* A *is commutative with trivial involution;*

(iii) B *is associative iff* A *is associative and commutative;*

(iv) B *is alternative iff* A *is alternative,* aa^* *commute with* A*, and* $2a + a^* \in N(A)$ *for all* $a \in A$*;*

(v) B *is flexible iff* A *is flexible,* aa^* *commute with* A*, and* $(a, b, c) = (a, b^*, c^*)$ *for all* $a, b, c \in A$*.*

[McC85] K. McCrimmon. “Nonassociative algebras with scalar involution”. In: *Pacific J. Math.* 116.1 (1985).

Let $C_*(A) := \{a \in C(A) \colon a^* = a\}$ and $Z_*(A) := \{a \in Z(A) \colon a^* = a\}$.

Theorem ([AB24])

If $B = A[X; *]^{\mathrm{fl}}$, then the following equalities hold:

(i) $C(B) = \{ \sum_{i\in\mathbb{N}} a_i X^i \colon a_i \in C_*(A), a_{2i+1}b^* = a_{2i+1}b \ \forall b \in A, i \in \mathbb{N}\}$;

(ii) $N_l(B) = N_r(B) = \{ \sum_{i\in\mathbb{N}} a_i X^i \colon a_{2i} \in Z(A), a_{2i+1} \in C(A) \cap N_m(A),$
$(a_{2i+1}b)c = a_{2i+1}(cb), (bc)a_{2i+1} = c(ba_{2i+1}) \ \forall b, c \in A, i \in \mathbb{N}\}$;

(iii) $N_m(B) = \{ \sum_{i\in\mathbb{N}} a_i X^i \colon a_{2i} \in C(A) \cap N_m(A), a_{2i+1} \in C(A),$
$(a_{2i+1}b)c = (a_{2i+1}c)b, b(ca_{2i+1}) = c(ba_{2i+1}) \ \forall b, c \in A, i \in \mathbb{N}\}$;

(iv) $N(B) = \{ \sum_{i\in\mathbb{N}} a_i X^i \colon a_i \in Z(A), a_{2i+1}[A, A] = 0 \ \forall i \in \mathbb{N}\}$;

(v) $Z(B) = \{ \sum_{i\in\mathbb{N}} a_i X^i \colon a_i \in Z_*(A), a_{2i+1}b^* = a_{2i+1}b \ \forall b \in A, i \in \mathbb{N}\}$.

Corollary ([AB24])

If $B = A[X;*]^{\mathrm{fl}}$, $1/2 \in K$, and A is obtained from K by n repeated applications of the Cayley-Dickson process, then the following equalities hold:

(i) $C(B) = Z(B) = \begin{cases} K[X] & \text{if } n = 0, \\ K[X^2] & \text{if } n \geq 1. \end{cases}$;

(ii) $Z_*(B) = K[X^2]$;

(iii) $N(B) = \begin{cases} B & \text{if } 0 \leq n \leq 1, \\ K[X^2] & \text{if } n \geq 2. \end{cases}$

Thank you!