Hom-Lie structure of generalized $\mathfrak{sl}(2)$-type

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Overview

Preliminaries

Preliminaries

We work over algebraically closed field $\mathbb{K}$ of characteristic zero. All vector spaces are over $\mathbb{K}$.

Definition 1 (Hom-Lie algebra)
A Hom-Lie algebra is a triple $(\mathscr{V}, [\cdot,\cdot], \alpha)$ consisting of a linear space $\mathscr{V}$, a bilinear map $[\cdot,\cdot] : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$, and a linear map $\alpha : \mathscr{V} \to \mathscr{V}$ satisfying the skew-symmetry condition and Hom-Jacobi identity properties. That is for all $x, y, z \in \mathscr{V}$:

(i) $[x, y] = -[y, x]$

(ii) $\sum_{\circlearrowleft_{x,y,z}} [\alpha(x), [y, z]] = 0.$

where $\circlearrowleft_{x,y,z}$ denotes summation over cyclic permutation on x, y, z.

For a skew-symmetric algebra $(\mathscr{V}, [\cdot,\cdot])$ by a Hom-Lie structure we mean the vector subspace of all linear twisting maps α that satisfies the Hom-Jacobi identity (ii).

Definition 2
Let $(\mathscr{V}_1, [\cdot,\cdot], \alpha)$ and $(\mathscr{V}_2, \{\cdot,\cdot\}, \beta)$ be Hom-Lie algebras. A Hom-Lie algebra morphism is a linear mapping

$$f : (\mathscr{V}_1, [\cdot,\cdot], \alpha) \to (\mathscr{V}_2, \{\cdot,\cdot\}, \beta)$$

satisfying the following conditions

(i) $f([x,y]) = \{f(x), f(y)\}$ for all $x, y \in \mathscr{V}_1$, and
(ii) $f \circ \alpha = \beta \circ f$.

When f satisfies only the first condition (i) we say that f is a weak morphism.

Definition 3

A Hom Lie-algebra $(\mathscr{V}, [\cdot,\cdot], \alpha)$ is said to be:

(i) Multiplicative if α is an algebra morphism.

(ii) Regular if α is an isomorphism.

Definition 4 (Hom-subalgebra)

Let $(\mathscr{V}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. A Hom-subalgebra is a subspace $\mathscr{W} \subseteq \mathscr{V}$ that is invariant under the linear α, and closed under bilinear bracket multiplication. That is:

(i) $\alpha(\mathscr{W}) \subseteq \mathscr{W}$, and

(ii) $[\mathscr{W}, \mathscr{W}] \in \mathscr{W}$.

Preliminaries

Definition 5 (Hom-Ideal)

Let $(\mathscr{V}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra. A Hom-ideal is a subspace $I \subseteq \mathscr{V}$ satisfying the following properties:

(i) $\alpha(I) \subseteq I$

(ii) $[\mathscr{V}, x] \in I$ for all $x \in I$.

Example 6

Let $\mathscr{A} = (\mathscr{V}, [\cdot,\cdot], \alpha)$ and $\mathscr{B} = (\mathscr{W}, \{\cdot,\cdot\}, \alpha)$ be Hom-Lie algebras and $f : \mathscr{A} \to \mathscr{B}$ be Hom-Lie algebra morphism and let $x \in \ker(f)$ and $v \in \mathscr{V}$ then,
$f(\alpha(x)) = \alpha(f(x)) = \alpha(0) = 0 \implies \ker(f)$ is α-invariant. Furthermore,

$$f([v,x]) = \{f(v), f(x)\} = \{f(v), 0\} = 0.$$

Therefore, $\ker(f)$ is a Hom-ideal.

Structure constants

Structure constants

Let $\{e_1, \dots, e_n\}$ be a basis for $\mathscr{V}$. The structure constants equation is given by:

$$[e_i, e_j] = \sum_{s=1}^{n} C_{i,j}^{s} e_s \quad \text{and} \quad \alpha(e_i) = \sum_{t=1}^{n} a_{ti} e_t, \text{ for all } C_{i,j}^{s}, a_{ti} \in \mathbb{K}. \tag{1}$$

The skew-symmetry and Hom-Jacobi identity can be rewritten as follows:

(i) **Skew-symmetry:** $[e_i, e_j] = -[e_j, e_i] \implies \sum_{s=1}^{n} C_{i,j}^{s} e_s = -\sum_{s=1}^{n} C_{j,i}^{s} e_s.$

(ii) **Hom-Jacobi identity**

$$\sum_{\circlearrowleft(i,j,k)} \left[\alpha(e_i), [e_j, e_k]\right] = \sum_{r=1}^{n} \left(\sum_{s=1}^{n} \sum_{t=1}^{n} \sum_{\circlearrowleft(i,j,k)} a_{ti} C_{j,k}^{s} C_{t,s}^{r} \right) e_r = 0.$$

Structure constants

This determines a subvariety of $\mathbb{K}^{\frac{n^2(n+1)}{2}}$ defined by the system of polynomial equations in (2), and is linear in a_{ti} variables

$$\sum_{s,t=1}^{n} \left(a_{ti}\left(C_{j,k}^{s}C_{t,s}^{r}\right)+a_{tk}\left(C_{i,j}^{s}C_{t,s}^{r}\right)+a_{tj}\left(C_{k,i}^{s}C_{t,s}^{r}\right)\right)=0, \tag{2}$$

for all $1 \leq i < j < k \leq n$ and $r = 1,2,\dots n$.

Equation (2) can be represented in matrix form as $\mathscr{M} a_\alpha = 0$, where $\mathscr{M}$ is a $\binom{n}{3} \cdot n \times n^2$ matrix and a_α is the column matrix. Therefore, the linear transformation $\mathscr{L}$ represented by $\mathscr{M}$ is

$$\mathscr{L} : \mathbb{K}^{n^2} \to \mathbb{K}^{\frac{n^2(n-1)(n-2)}{6}}. \tag{3}$$

For a multiplicative Hom-Lie algebras the weak morphism condition can be written as follows: For all $1 \leq i < j \leq n$,

$$\sum_{s=1}^{n}\sum_{r=1}^{n} a_{rs}C_{i,j}^{s}e_r - \sum_{t=1}^{n} a_{ti} \sum_{p=1}^{n} a_{pj} \sum_{q=1}^{n} C_{t,p}^{q} e_q = 0. \tag{4}$$

3-dimensional Hom-Lie Algebras

3-dimensional Hom-Lie Algebras

Let $\{e_1, e_2, e_3\}$ be a basis for $\mathscr{V}$ then, the Hom-Jacobi identity given by the system of polynomial Equations (2) become

$$\begin{aligned}
&a_{11}\left(C_{2,3}^{3}C_{1,3}^{r}+C_{2,3}^{2}C_{1,2}^{r}\right)+a_{12}\left(-C_{1,3}^{3}C_{1,3}^{r}-C_{1,3}^{2}C_{1,2}^{r}\right)+a_{13}\left(C_{1,2}^{3}C_{1,3}^{r}+C_{1,2}^{2}C_{1,2}^{r}\right)+\\
&+a_{21}\left(C_{2,3}^{3}C_{2,3}^{r}-C_{2,3}^{1}C_{1,2}^{r}\right)+a_{22}\left(C_{1,3}^{1}C_{1,2}^{r}-C_{1,3}^{3}C_{2,3}^{r}\right)+a_{23}\left(C_{1,2}^{3}C_{2,3}^{r}-C_{1,2}^{1}C_{1,2}^{r}\right)+\\
&+a_{31}\left(-C_{2,3}^{1}C_{1,3}^{r}-C_{2,3}^{2}C_{2,3}^{r}\right)+a_{32}\left(C_{1,3}^{2}C_{2,3}^{r}+C_{1,3}^{1}C_{1,3}^{r}\right)+a_{33}\left(-C_{1,2}^{1}C_{1,3}^{r}-C_{1,2}^{2}C_{2,3}^{r}\right)=0
\end{aligned} \tag{5}$$

for all $r = 1,2,3$ and the linear transformation $\mathscr{L}$ represented by the matrix $\mathscr{M}$ in (3) is given by: $\mathscr{L} : \mathbb{K}^9 \to \mathbb{K}^3$.

For the multiplicative Hom-Lie algebras the system of polynomial equations in (4) become:

$$\begin{aligned}
&(a_{11}C_{i,j}^{1}+a_{12}C_{i,j}^{2}+a_{13}C_{i,j}^{3})e_1+(a_{21}C_{i,j}^{1}+a_{22}C_{i,j}^{2}+a_{23}C_{i,j}^{3})e_2+(a_{31}C_{i,j}^{1}+a_{32}C_{i,j}^{2}+a_{33}C_{i,j}^{3})e_3\\
&\quad=(a_{3i}a_{2j}C_{3,2}^{1}+a_{2i}a_{3j}C_{2,3}^{1})e_1+(a_{2i}a_{1j}C_{2,1}^{2}+a_{1i}a_{2j}C_{1,2}^{2})e_2+(a_{3i}a_{1j}C_{3,1}^{3}+a_{1i}a_{3j}C_{1,3}^{3})e_3
\end{aligned} \tag{6}$$

For all $i, j = 1,2,3$.

3-dimensional Hom-Lie Algebras

Consider skew-symmetric algebra $(\mathscr{V}, [\cdot,\cdot])$. We aim at constructing all the linear space of all linear twisting maps α such that the bracket multiplication defined by

$$[e_1, e_2] = \lambda_1 e_2, [e_1, e_3] = \lambda_2 e_3, [e_2, e_3] = \lambda_3 e_1, \forall\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{K} \tag{7}$$

determines 3-dimensional Hom-Lie algebra, and provide all classes and subclasses of this Hom-Lie algebra.

Example 7

When $\lambda_1, \lambda_2, \lambda_3 \neq 0$ then, the solution set of (5) is given by

$$\{a_{ij}, 1 \leq i, j \leq 3 \mid a_{33} = -\frac{\lambda_2}{\lambda_1} a_{22}, a_{21} = \frac{\lambda_1}{\lambda_3} a_{13}, a_{31} = -\frac{\lambda_2}{\lambda_3} a_{12}\}$$

and the resulting Hom-Lie structure is of dimension 6 given by the matrix

$$[\alpha] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{\lambda_1}{\lambda_3} a_{13} & a_{22} & a_{23} \\ -\frac{\lambda_2}{\lambda_3} a_{12} & a_{32} & -\frac{\lambda_2}{\lambda_1} a_{22} \end{pmatrix}.$$

Furthermore, solving (5) and (6) simultaneously we obtain the following subsets of the linear space of linear twisting maps α.

(a) When $\lambda_1 \neq \pm\lambda_2$ for all $\lambda_1, \lambda_2, \lambda_3 \neq 0$, $[\alpha] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & -\frac{\lambda_2}{\lambda_1} a_{22} \end{pmatrix}$, where $a_{22}^2 = -\frac{\lambda_1}{\lambda_2}$

(b) When $\lambda_1 = -\lambda_2$ for all $\lambda_1, \lambda_2, \lambda_3 \neq 0$, then, α is given by:

(i) $\begin{pmatrix} a_{11} & \frac{(-1+a_{11})(1+a)\lambda_3}{2a_{13}\lambda_2} & a_{13} \\ \frac{\lambda_1}{\lambda_3} a_{13} & -\frac{1}{2}(1+a_{11}) & -\frac{a_{13}\lambda_2}{(1+a_{11})\lambda_3} \\ \frac{(1-a_{11})(1+a_{11})}{2a_{13}} & \frac{(-1+a_{11})^2(1+a_{11})\lambda_3}{4a_{13}\lambda_2} & -\frac{1}{2}(1+a_{11}) \end{pmatrix}$, for $a_{13} \neq 0, a_{11} \neq -1$

(ii) $\begin{pmatrix} 1 & a_{12} & 0 \\ 0 & -1 & 0 \\ -\frac{\lambda_2}{\lambda_3} a_{12} & -\frac{a_{12}^2 \lambda_2}{\lambda_3} & -1 \end{pmatrix}$ (iii) $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{32}} \\ 0 & a_{32} & 0 \end{pmatrix}$ for $a_{32} \neq 0$ (iv) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) When $\lambda_1 = \lambda_2$, for all $\lambda_1, \lambda_2, \lambda_3 \neq 0$ then, α is given by:

(i) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & -\frac{1+a_{22}^2}{a_{32}} \\ 0 & a_{32} & -a_{22} \end{pmatrix}$ for $a_{32} \neq 0$

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & -a_{22} \end{pmatrix}$ where $a_{22}^2 = -1$.

Lemma 8

Multiplicative Hom-Lie algebras arising from simple Lie algebras of $\mathfrak{sl}(2)$-type with non-zero twisting maps are regular Hom-Lie algebras.

Solvability and Nilpotency of Hom-Lie Algebras

Solvability and Nilpotency of Hom-Lie Algebras

We explore more properties of Hom-Lie algebras through its derived series and central descending series of an ideal I in $\mathscr{V}$.

Definition 9

Let $(\mathscr{V}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra, and $I \subseteq \mathscr{V}$ be an ideal of $\mathscr{V}$.

(i) A derived series of an ideal I is defined as

$$D^0(I) = I \text{ and } D^{k+1}(I) = [D^k(I), D^k(I)], \tag{8}$$

(ii) A central descending series of I is defined as

$$C^0(I) = I \text{ and } C^{k+1}(I) = \left[C^k(I), I\right]. \tag{9}$$

Furthermore, I is said to be solvable (resp. nilpotent) of at most class k if $D^k(I) = \{0\}$ (resp. $C^k(I) = \{0\}$) and $D^{k-1}(I) \neq \{0\}$ (resp. $C^{k-1}(I) \neq \{0\}$) for some $k \in \mathbb{Z}_{>0}$.

Lemma 10

Let $(\mathscr{V},[\cdot,\cdot],\alpha)$ be a Hom-Lie algebra, and I be an ideal of $\mathscr{V}$ for some $k\in\mathbb{Z}_{\geq 0}$, we have the following:

(i) $D^{k+1}(I)\subseteq D^{k}(I)$,

(ii) $C^{k+1}(I)\subseteq C^{k}(I)$.

Lemma 11

Let $\mathscr{A}=(\mathscr{V},[\cdot,\cdot])$ be a skew-symmetric algebra.

(i) *If $\mathscr{A}$ is nilpotent, then $Z(\mathscr{A})$ is not trivial.*

(ii) *If $\dim\ \mathscr{A}=3$, then $\dim\ Z(\mathscr{A})=0$ or $\dim\ Z(\mathscr{A})=1$ or $Z(\mathscr{A})=\mathscr{A}$.*

Proposition 12

Let $\mathscr{A}=(\mathscr{V},[\cdot,\cdot],\alpha)$ be 3-dimensional Hom-Lie algebra, then $\mathscr{A}$ is class 2 nilpotent if and only if $\dim Z(\mathscr{A})=1$ and $C^{1}(\mathscr{A})=Z(\mathscr{A})$.

Corollary 13

Let $\mathscr{A}$ be a skew-symmetric algebra:

(1) *$\mathscr{A}$ is nilpotent of class at most r if and only if for all $1 \leq k_0, \ldots, k_r \leq n, 1 \leq t_0 \leq n$,*

$$\sum_{t_1=1}^{n} \cdots \sum_{t_{r-1}=1}^{n} \Big(\prod_{j=0}^{r-2} C_{k_j, t_{j+1}}^{t_j} \Big) C_{k_{r-1}, k_r}^{t_{r-1}} = 0. \tag{10}$$

(2) *$\mathscr{A}$ is solvable of class at most r if and only if for all $1 \leq k_1, \ldots, k_{2^r} \leq n$,*

$$\sum_{t_0=1}^{n} \cdots \sum_{t_{2^r-3}=1}^{n} \Big(\prod_{j=0}^{2^{r-1}-1} C_{k_{2j+1}, k_{2j+2}}^{t_j} \Big) \Big(\prod_{p=0}^{2^{r-1}-2} C_{t_{2p}, t_{2p+1}}^{t_{2^{r-1}+p}} \Big) = 0. \tag{11}$$

Remark 14

Let $\mathscr{A}$ be a skew-symmetric algebra. If $\mathscr{A}$ is nilpotent (resp. solvable) of class at most r then, the resulting algebraic subvariety in $(C_{i,j}^k)_{i<j}$ variables is determined by n^{r+2} (resp. n^{2^r+1}) homogeneous polynomial equations of degree r, (resp. 2^r-1).

Lemma 15

Let $\mathscr{A}=(\mathscr{V},[\cdot,\cdot])$ be n-dimensional skew-symmetric algebra and $\alpha\in\mathrm{End}(\mathscr{V})$ be a Hom-Lie structure on $\mathscr{V}$. If $\mathscr{A}$ is class 2 nilpotent, then linear space of Hom-Lie structures is of dimension n^2.

Example 16

Consider 3-dimensional skew-symmetric algebra $\mathscr{A}=(\mathscr{V},[\cdot,\cdot])$ with the bracket multiplication defined as $[e_2,e_3]=\lambda e_1$. Then, $C^1(\mathscr{V})=Z(\mathscr{V})=\mathrm{span}\{e_1\}$. Therefore, $\mathscr{A}$ is class 2 nilpotent. Moreover, the linear space of Hom-Lie structure is of dimension 9.

Hom-subalgebras and Hom-ideals

Hom-subalgebras and Hom-ideals

We present some classes of non-multiplicative Hom-Lie algebras with the properties that their derived series and central descending series are α-invariant. We give instances when these derived algebras are Hom-subalgebras and Hom-ideals.

Proposition 17

Let $\mathscr{A} = (\mathscr{V}, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra, and let $I \subseteq \mathscr{V}$ be a Hom-ideal.

(i) *Then $D^1(I)$ and $C^1(I)$ are weak-ideals of $\mathscr{A}$.*

(ii) *For all $n \in \mathbb{Z}_{\geq 0}$, $D^n(I)$, and $C^n(I)$ are weak subalgebras of $\mathscr{V}$ and if in addition, $\mathscr{A}$ is multiplicative, then the subspaces $D^n(I)$ and $C^n(I)$ are Hom-subalgebras of $\mathscr{A}$.*

Proposition 18

Let $(\mathscr{V}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie algebra and I be Hom-ideal of $\mathscr{V}$. If the linear map α is surjective then, for all $n \in \mathbb{Z}_{\geq 0}$, $D^n(I)$ and $C^n(I)$ are Hom-ideals.

Table 1: The linear space of Hom-Lie structures of generalized Hom-Lie algebras of $\mathfrak{sl}(2)$-type, multiplicative Hom-Lie algebras, Hom-ideals, and solvability and nilpotency properties.

<table>
<tr><th>Structure Constants</th><th>$[\alpha]$ such that $(\mathscr{V},[\cdot,\cdot],\alpha)$ is a Hom-Lie algebra</th><th>Derived Series, Central descending series of $\mathscr{V}$</th><th>$[\alpha]$ such that $C^r(\mathscr{V})$, $D^r(\mathscr{V})$ are α-invariant</th><th>1-dim weak ideal and $[\alpha]$ that turn them into Hom-ideals</th><th>$[\alpha]$ such that $(\mathscr{V},[\cdot,\cdot],\alpha)$ is multiplicative</th></tr>
<tr><td rowspan="3">$\lambda_1=0$
$\lambda_2,\lambda_3\neq 0$</td><td rowspan="3">$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ -\frac{\lambda_2}{\lambda_3}a_{12} & a_{32} & a_{33} \end{pmatrix}$</td><td>Not nilpotent
Class-3-Solvable</td><td rowspan="2">$a_{21}=a_{23}=0$
$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ -\frac{\lambda_2}{\lambda_3}a_{12} & a_{32} & a_{33} \end{pmatrix}$</td><td rowspan="3"></td><td rowspan="3">$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_{32} & 0 \end{pmatrix}$</td></tr>
<tr><td>For all $r\geq 1$
$C^r(\mathscr{V})=\text{span}\{e_1,e_3\}$
$=D^1(\mathscr{V})$
Weak ideal</td></tr>
<tr><td>$D^2(\mathscr{V})=\text{span}\{e_3\}$
Not weak ideal</td><td>$a_{13}=a_{23}=0$
$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & 0 \\ -\frac{\lambda_2}{\lambda_3}a_{12} & a_{32} & a_{33} \end{pmatrix}$</td></tr>
</table>

$\lambda_2=0$ $\lambda_1,\lambda_3\neq 0$	$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{\lambda_1}{\lambda_3}a_{13} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix}$	Not nilpotent Class-3-Solvable			$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$
		For $r\geq 1$ $C^r(\mathscr{V})=\text{span}\{e_1,e_2\}$ $=D^1(\mathscr{V})$ Weak ideal	$a_{31}=a_{32}=0$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \frac{\lambda_1}{\lambda_3}a_{13} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$		
		$D^2(\mathscr{V})=\text{span}\{e_2\}$ Not weak ideal	$a_{12}=a_{32}=0$ $\begin{pmatrix} a_{11} & 0 & a_{13} \\ \frac{\lambda_1}{\lambda_3}a_{13} & a_{22} & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix}$		
$\lambda_3=0$ $\lambda_1,\lambda_2\neq 0$	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	Not nilpotent Class-2-Solvable			
		For $r\geq 1$ $C^r(\mathscr{V})=\text{span}\{e_2,e_3\}$ $=D^1(\mathscr{V})$ Hom-ideal	$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\text{span}\{c_2e_2+c_3e_3\}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & \mathfrak{k} \end{pmatrix}$ where $\mathfrak{k}=a_{22}+\frac{c_3}{c_2}a_{23}-\frac{c_2}{c_3}a_{32}$ $\text{span}\{e_2\}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$ $\text{span}\{e_3\}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\lambda_1\neq\pm\lambda_2$ $\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{\lambda_1}{\lambda_2} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\begin{pmatrix} \frac{\lambda_2}{\lambda_1} & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & 0 \end{pmatrix}$

					$\lambda_1=\lambda_2$ $\begin{pmatrix}1&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ $\begin{pmatrix}1&0&0\\a_{21}&a_{22}&0\\a_{31}&a_{32}&a_{33}\end{pmatrix}$ $\begin{pmatrix}1&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&0&a_{33}\end{pmatrix}$ $\begin{pmatrix}1&0&0\\a_{21}&0&a_{23}\\a_{31}&a_{32}&0\end{pmatrix}$ $\lambda_1=-\lambda_2$ $\begin{pmatrix}-1&0&0\\a_{21}&0&a_{23}\\a_{31}&a_{32}&0\end{pmatrix}$
$\lambda_1,\lambda_2=0$ $\lambda_3\neq 0$	$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$	Class-2-nilpotent Class-2-Solvable $C^1(\mathscr{V})=\text{span}\{e_1\}$ $=D^1(\mathscr{V})$ Weak ideal	$a_{21}=a_{31}=0$ $\begin{pmatrix}a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&a_{32}&a_{33}\end{pmatrix}$	$\text{span}\{e_1\}$ $\begin{pmatrix}a_{11}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&a_{32}&a_{33}\end{pmatrix}$	$\begin{pmatrix}a_{22}a_{33}-a_{23}a_{32}&a_{12}&a_{13}\\0&a_{22}&a_{23}\\0&a_{32}&a_{33}\end{pmatrix}$

$\lambda_1,\lambda_3=0$ $\lambda_2\neq 0$	$\begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	Not nilpotent Class-2-Solvable For $r\geq 1$ $C^r(\mathscr{V})=\text{span}\{e_3\}$ $=D^1(\mathscr{V})$ Weak ideal	$a_{13}=a_{23}=0$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\text{span}\{e_2\}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$ $\text{span}\{e_3\}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$
$\lambda_2,\lambda_3=0$ $\lambda_1\neq 0$	$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	Not nilpotent Class-2-Solvable $C^1(\mathscr{V})=\text{span}\{e_2\}$ $=D^1(\mathscr{V})$ Weak ideal	$a_{12}=a_{32}=0$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$	$\text{span}\{e_2\}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$ $\text{span}\{e_3\}$ $\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$ $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$

Remarks

(i) In both cases 1 and 2, the triple $(\mathscr{V}, [\cdot,\cdot], \alpha)$ defines a Hom-Lie algebra. However, the bilinear bracket multiplication do not define a Lie algebra.

(ii) In Case 3, we note that this algebra not Hom-simple in general, because there exists a Hom-ideal $I = \mathrm{span}\{e_2, e_3\}$.

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Thank you!