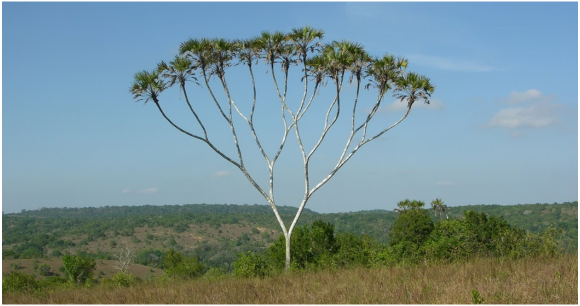
Strange algebra identities

Vladimir G. Tkachev



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Preliminaries

- $\mathbf{K}$ is a field of characteristic not equal to 2 and 3;
- $\mathbb{A}$ is a commutative but nonassociative algebra with multiplication $\bullet$ over $\mathbf{K}$
- $L(x)y := x \bullet y$;
- $\mathrm{Idm}(\mathbb{A})$ is the set of nonzero algebra idempotents;
- given $x \in \mathbb{A}$, $\langle\langle x \rangle\rangle$ denotes the **subalgebra** generated by x;
- a nonasociative monomial x^α is an element of the **multiplicative magma** $\langle\langle x \rangle\rangle$;
- there is a natural **grading** on nonaasociative monomials x^α: $\deg : x^\alpha \to \mathbb{Z}^+$;
- a linear combination of x^α is a nonassociative **polynomial** P in x;
- $P = 0$ is an **identity** on $\mathbb{A}$ if $P(x) = 0$ for any $x \in \mathbb{A}$;

Remark

One of the most interesting features of the concepts considered below is their applications to study of fusion rules, but we don't consider this issue in our present talk. A part of material below is based on my paper *The universality of one half in commutative nonassociative algebras with identities.* J. Algebra, 569, 466-510, 2021.

Preliminaries

There are two special nonassociative monomials parametrized by $\mathbb{N}$:

- **principal** powers: $x^0 := 1 \in \mathbf{K}$, $x^1 := x$ and $x^n := x \bullet x^{n-1}$, $n \geq 2$;
- **plenary** powers: $x^{[1]} := x$ and $x^{[n]} := x^{[n-1]} \bullet x^{[n-1]}$, $n \geq 2$;

Observe that

$$\deg x^n = n, \qquad \deg x^{[n]} = 2^{n-1}.$$

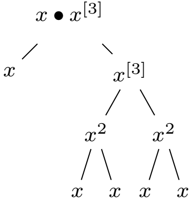
For example,

$$x, \quad x^2 = x \bullet x, \quad x^3 = (x \bullet x) \bullet x, \qquad x^{[3]} = (x \bullet x) \bullet (x \bullet x), \ldots$$

A general monomial can have a very involved structure, for example

$$(x \bullet x) \bullet ((((x \bullet x) \bullet x) \bullet x) \bullet (x \bullet x)) = x^{[2]} \bullet (x^4 \bullet x^{[2]}).$$

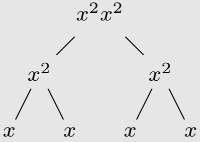
It is natural to represent monomials by binary trees, for example



There exists a one-to-one correspondence between

- nonassociative monomials x^α of degree n;
- **parenthesizations** on n symbols,
- (labeled full) **binary trees** with n leaves,
- **path-length sequences** $[k_1, \dots, k_m]$ such that $\sum_{i=1}^{m} \frac{1}{2^{k_i}} = 1$. Here, k_i is the number of paths connecting the ith leaf with the parent.

For example, $x^2 \bullet x^2 = (x \bullet x) \bullet (x \bullet x)$ is represented by the tree



and the tree's **path-length sequence** $[2, 2, 2, 2]$, where $\sum_{i=1}^{4} \frac{1}{2^2} = 1$.

Proposition [Knuth, 2.3.4.5, M24]

Given any sequence $[k_1, \dots, k_m]$ such that $\sum_{i=1}^{m} \frac{1}{2^{k_i}} = 1$ there exists a binary tree such that $[k_1, \dots, k_m]$, maybe **reordered**, is its path-length sequence.

Given a multiset A of numbers we denote by $|A| = \sum_{a \in A} a$.

Lemma 1

(i) Let A be a multiset of cardinality at least 2 consisting of nonnegative powers of 2, and such that $|A| = 2^L$ for some integer L. Then one can split $A = A' \sqcup A''$ into two subsets such that $|A'| = |A''| = 2^{L-1}$.
(ii) Let A be a multiset of cardinality at least 2 consisting of **nonpositive** *powers of 2, and such that $|A| = 1$. Then one can split $A = A' \sqcup A''$ into two subsets such that $|A'| = |A''| = \frac{1}{2}$.*

Corollary 2

Given any sequence $[k_1, \ldots, k_m]$ such that $\sum_{i=1}^{m} \frac{1}{2^{k_i}} = 1$ there exists a binary tree such that $[k_1, \ldots, k_m]$, maybe **reordered***, is its path-length sequence.*

Proof. (i) Let $A' \subset A$ be a proper subset such that $|A'| < 2^{L-1}$. If $A'' = A \setminus A'$ then

$$\Delta := 2^{L-1} - |A'| = |A''| - 2^{L-1} = (2^{x_1} + \ldots + 2^{x_m}) - 2^{L-1}$$

where the RHS is divisible by the smallest power 2^{x_m}. Then $\Delta \geq 2^{x_m}$, hence sending 2^{x_m} to the LHS obtain

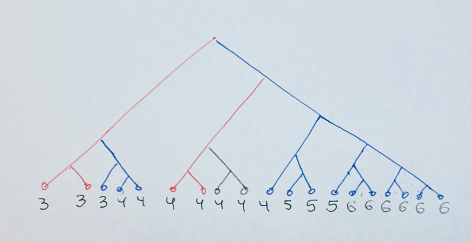
$$|A' \sqcup \{2^{x_m}\}| = |A'| + 2^{x_m} = (2^{L-1} - \Delta) + 2^{x_m} \leq 2^{L-1}.$$

Repeating the process implies the desired conclusion in (i).

$A = \{(\frac{1}{2^6})^{\#6}, (\frac{1}{2^5})^{\#3}, (\frac{1}{2^4})^{\#7}, (\frac{1}{2^3})^{\#3}\}$ satisfies $|A| = \frac{6}{2^6} + \frac{3}{2^5} + \frac{7}{2^4} + \frac{3}{2^3} = 1$. A particular solution

$$\underbrace{\underbrace{\frac{1}{2^3}+\frac{1}{2^3}}_{=\frac{1}{4}}+\underbrace{\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^4}}_{=\frac{1}{4}}}_{=\frac{1}{2}} = \underbrace{\underbrace{\frac{1}{2^4}+\frac{1}{2^4}+\frac{1}{2^4}+\frac{1}{2^4}}_{=\frac{1}{4}}+\underbrace{\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^5}+\frac{1}{2^5}+\frac{1}{2^6}+\ldots+\frac{1}{2^6}}_{=\frac{1}{4}}}_{=\frac{1}{2}}$$

corresponds to the binary tree $[3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, 6]$



Observe that one can shuffle some ‘branches’ such that the resulting tree will be different but still have the same unordered path-length sequence. It is natural to call such trees by **isomers**.

So what is this all about and why?...

The Peirce polynomial

Example 1

By definition, an algebra is power-associative if any subalgebra $\langle\langle x\rangle\rangle$ is associative. In the commutative case this is equivalent to that the following identity holds:

$$P(x) := x^4 - x^2 \bullet x^2 = 0 \tag{1}$$

Indeed, the linearization yields

$$2L(x)^3 + L(x)L(x^2) - 4L(x^2)L(x) + L(x^3) = 0 \tag{2}$$

and conversely, (2) implies (1). Next, by (1): $(x^2 \bullet x^2) \bullet x = x^5$ and applying (2) to x^2 gives $x^2 \bullet x^3 = x^5$. This implies the desired conclusion by induction. If $c \in \mathrm{Idm}(\mathbb{A})$ then (2) becomes

$$\underbrace{2L(c)^3 - 3L(c)^2 + L(c)}_{\text{the \textbf{Peirce polynomial}}} = \underbrace{(2L(c)-1)(L(c)-1)L(c) = 0}_{\text{implies the \textbf{Peirce spectrum}}} \tag{3}$$

This yields the corresponding **Peirce decomposition**:

$$\mathbb{A} = \mathbb{A}_1 \oplus \mathbb{A}_0 \oplus \mathbb{A}_{\frac{1}{2}}.$$

The Peirce polynomial

Example 2.

A **Bernstein** algebra is a commutative algebra with a nontrivial algebra homomorphism $\omega : \mathbb{A} \to \mathbf{K}$ satisfying the following (genetic type) identity:

$$x^2 \bullet x^2 - \omega(x)^2 x^2 = 0. \tag{4}$$

Then linearization yields $4x^2 \bullet (x \bullet y) - 2\omega(x)^2 x \bullet y - 2\omega(x)\omega(y)x^2 = 0$, hence

$$4L(x^2)L(x) - 2\omega(x)^2 L(x) - 2\omega(x)x^2 \otimes \omega^* = 0.$$

If $x = c$ is a nonzero idempotent then $\omega(c) = 1$ and

$$4L(c)^2 - 2L(c) - 2c \otimes \omega^* = 0.$$

In particular, if x is an eigenvector of $L(c)$ then

$$\underbrace{(4L(c)^2 - 2L(c))}_{\text{the Peirce polynomial}} x = \underbrace{2c \otimes \omega^*}_{\text{must be zero}} = 0.$$

Some further examples of binomial identities

- **Quasicomposition** algebras:

$$x^3 - Q(x)x = 0, \tag{5}$$

  where Q is a quadratic form.

- The **Elduque-Okubo** algebras:

$$x^2 \bullet x^2 - N(x)x = 0, \tag{6}$$

  where $N(x)$ is a cubic form.

- The **Bernstein** algebras:

$$x^2 \bullet x^2 - \omega(x)^2 x^2 = 0, \tag{7}$$

  where $\omega : \mathbb{A} \to \mathbf{K}$ is a homomorphism.

- A **strange** identity:

$$(x^3)^2 - (x^2)^3 = 0. \tag{8}$$

Observations: algebras satisfying (8):

- any algebra with $\mathbb{A}^3 = 0$;
- any power-associative algebra (in particular, any Jordan algebra);
- any medial algebra (indeed, $(x \bullet x) \bullet (x^2 \bullet x^2) = (x \bullet x^2) \bullet (x \bullet x^2)$).

The Peirce polynomial

$f(x)$ is a homogeneous polynomial function of order m if $\Delta_m f = 0$ and $f(\lambda x) = \lambda^m f(x)$.

We consider a commutative algebra $\mathbb{A}$ satisfying a homogeneous identity of the kind

$$P := \sum_{\alpha} \phi_\alpha(x) x^\alpha = 0 \tag{9}$$

where $\phi_\alpha(x)$ are homogeneous polynomial functions. To any such polynomial one associate the so-called Perice polynomial (linearization evaluated at an idempotent c). More precisely,

$$\underbrace{D_y(P(x))}_{\text{at } c} = \varrho_c(P, L(c)) + {\color{red} c \otimes (\ldots)}$$

where $\varrho_c(P, L(c))$ is a certain polynomial in $L(c)$ over $\mathbf{K}$. In particular,

$$\varrho_c(P, L(c)) = 0 \qquad \text{on any } \textbf{proper} \text{ eigensubspace of } L(c)$$

Definition. Given an idempotent c, the **Peirce polynomial** $\varrho_c(P; q) \in \mathbf{K}[q]$ is defined by

$$\varrho_c(\sum_{\alpha} \phi_\alpha(x) x^\alpha, q) := \sum_{\alpha} \phi_\alpha(c) \rho(x^\alpha, q)$$

where ρ is the Peirce operator defined below.

(i) Given a formal indeterminate q, the **Peirce operator**

$$\rho : \langle\langle x \rangle\rangle \to \mathbb{Z}[q]$$

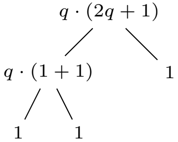
is uniquely determined by $\rho(x, q) = 1$ and the recurrence relation

$$\rho(x^{\alpha} x^{\beta}, q) = q \cdot (\rho(x^{\alpha}, q) + \rho(x^{\beta}, q)). \tag{10}$$

(ii) Alternatively, by means of the tree's **path-length sequence** one has an explicit expression

$$\rho([k_1, k_2, \ldots, k_m], q) = \sum_{i=1}^{m} q^{k_i}.$$

For example, $\rho(x^3, q) = \rho([1, 2, 2], q) = q^1 + q^2 + q^2 = 2q^2 + q$, or



By Corollary 2 (or by induction), for any binary tree one has

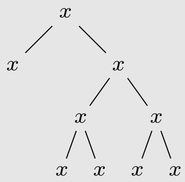
$$\boxed{\rho(x^{\alpha}, \frac{1}{2}) = \sum_{i=1}^{m} \frac{1}{2^{k_i}} = 1}$$

Some further examples:

$$\begin{aligned}
\rho(x^2, q) &= 2q, \\
\rho(x^3, q) &= 2q^2 + q, \\
\rho(x^4, q) &= 2q^3 + q^2 + q, \\
\rho(x^2 \bullet x^2, q) &= 4q^2, \\
\rho((x^2 \bullet x^2) \bullet x, q) &= 4q^3 + q.
\end{aligned} \tag{11}$$

Remark. Given a Peirce polynomial, the corresponding **unordered** path-length sequence can be restored, for example,

$$q + 4q^3 \quad \to \quad q + \underbrace{q^3 + \ldots + q^3}_{\text{4 times}} \quad \to \quad [1, 3, 3, 3, 3] \quad \to \quad x \bullet (x^2 \bullet x^2)$$



Corollary 2 implies

Proposition 1

For any nonassiciative monomial x^α, $P(x^\alpha, q) \in \mathbb{Z}^+[q]$ *and* $P(x^\alpha, \frac{1}{2}) = 1$. *Conversely, if* $Q \in \mathbb{Z}^+[q]$ *and* $Q(\frac{1}{2}) = 1$ *then there exists a monomial* x^α *such that* $Q(q) = P(x^\alpha, q)$.

Remarks

- There is no Peirce polynomial of the kind

$$P(x^\alpha, q) = 3q^3 + \textit{lower degree terms}.$$

  Indeed, if $P(x^\alpha, q) = 3q^3 + aq^2 + bq$ (the constant term must be zero because $P(x^\alpha, \frac{1}{2}) = 1$), then $2a + 4b = 5$, a contradiction.

- In general, the leading coefficient must be even. For example, it easy to see that $P(x^\alpha, q) = 6q^3 + \textit{lower degree terms}$ implies $6q^3 + q^2$ and uniquely determines $x^\alpha = (x^2 \bullet x^2) \bullet x^3$.

- Isomeric binary trees have the same Peirce polynomial, for example $(x^3)^2$ and $(x^2)^3$.

The multiplication of two monomials $x^\alpha \bullet x^\beta$ imply the operation of merging of the corresponding Peirce polynomials $P \star Q$.

It follows from the definition (10) that this merging (fusion) is an isotopy of the standard associative addition, therefore the resulting operation is a medial magma. More precisely:

The set of polynomials

$$\Omega := \{Q \in \mathbb{Z}^+[q] : \ Q(\frac{1}{2}) = 1\}$$

is a **medial magma** with respect to

$$P \star Q := q \cdot (P + Q).$$

Indeed,

$$(P \star Q) \star (R \star T) = q^2(P + Q + R + T)$$

The universality of $\frac{1}{2}$

Theorem 3 (V.T., *J. Algebra, 2021*)

Let a commutative algebra $\mathbb{A}$ *satisfy* $P := \sum_\alpha \phi_\alpha(x)x^\alpha = 0$ *and* $c \in \mathrm{Idm}(\mathbb{A})$*. Then*

$$\rho(P, \tfrac{1}{2}) = 0$$

If additionally c *is semi-simple and* λ *is a simple root of the Peirce polynomial* $\varrho_c(P, t)$ *then*

$$\mathbb{A}_c(\lambda)\mathbb{A}_c(\tfrac{1}{2}) \subset \bigoplus_{\nu\in\sigma(c),\nu\neq\lambda} \mathbb{A}_c(\nu). \tag{12}$$

Example. For power-associative algebra identity one obtains (cf. with (3))

$$\rho(x^4 - x^2 \bullet x^2, q) = (2q^3 + q^2 + q) - 4q^2 = q(q-1)(2q-1).$$

Binomials

Let us consider homogeneous binomials, i.e. nonassociative polynomials with two terms with coefficients in $\mathbf{K}$. If such an algebra contains a nonzero idempotent, it is necessarily that the identity has the form

$$x^\alpha - x^\beta = 0, \qquad \deg x^\alpha = \deg x^\beta.$$

Except of (1) in degree 4, there are $\binom{3}{2} = 3$ *binomials* in degree 5

$$\begin{aligned} S_1 &:= x^5 - (x^2 \bullet x^2) \bullet x \\ S_2 &:= x^5 - x^2 \bullet x^3, \\ S_3 &:= (x^2 \bullet x^2) \bullet x - x^2 \bullet x^3 = S_2 - S_1, \end{aligned}$$

which imply respectively the corresponding Peirce polynomials

$$\begin{aligned} \rho(S_1, q) &= (2q^4 + q^3 + q^2 + q) - (4q^2 + 1)q = q^2(q-1)(2q-1) \\ \rho(S_2, q) &= (2q^4 + q^3 + q^2 + q) - (2q^2 + 3q)q = q(q-1)(q+1)(2q-1) \\ \rho(S_3, q) &= q(q-1)(2q-1). \end{aligned}$$

Degenerate identities

But in degree 6, for the first time appears an nonzero identity with zero Peirce polynomial,

$$\boxed{R := (x^3)^2 - (x^2)^3 = 0.} \tag{13}$$

Indeed,

$$\rho((x^3)^2, q) = 2\rho(x^3, q) \cdot q = 2(2q^2 + q)q$$
$$\rho((x^2)^3, q) = (\rho((x^2)^2, q) + \rho(x^2, q)) \cdot q = (4q^2 + 2q)q.$$

implies $\rho(R, q) = 0$.

Remarks.

- In [V.T., 2021] we referred to such identities as **degenerated identities**. In particular this implies that the Peirce spectrum is undetermined for any algebra idempotent.
- In [Varro, 2020]: such identities are called **evanescent**. Varro studies these in the context of baric train algebras.

Decomposable degenerate identities

Let ω be a baric homomorphism on $\mathbb{A}$ and suppose $\mathbb{A}$ satisfy

$$P := x^2 \bullet x^2 - 2\omega(x)x^3 + \omega(x)^2 x^2 = 0.$$

Then P is degenerated: $\rho(P, q) = 0$. Algebras with $P = 0$ were studied in [Ouattara-Zitan et al., J. Alg., 2010] and later by Elduque and Labra [J. Alg. Appl,, 2013], where it was proven that the 'gametization' of the original multiplication in $\mathbb{A}$ to $x \star y = xy - \frac{1}{2}\omega(x)y - \frac{1}{2}\omega(y)x$ satisfies the plenary nilpotent identity $x^{\star 2} \star x^{\star 2} = 0$, while $\mathbb{A}^{\star 2} \neq \mathbb{A}$. Note that P is decomposable:

$$P = (x^2 - \omega(x)x)^2.$$

Theorem (V.T., 2021)

Let $\mathbb{A}$ be an algebra with identity $P(z) = 0$ such that $P = P_1 \bullet P_2$ is decomposable in the free nonassociative algebra $\mathbf{K}(\langle\langle x \rangle\rangle)$. Then for any nonzero idempotent c of A there holds

$$\varrho_c(P_1, \tfrac{1}{2})\, \varrho_c(P_2, \tfrac{1}{2}) = 0.$$

Furthermore,

- if $\varrho_c(P_1, \frac{1}{2}) = \varrho_c(P_2, \frac{1}{2}) = 0$ then P is a degenerated identity;
- if $\varrho_c(P_i, \frac{1}{2}) = 0$ and $\varrho_c(P_j, \frac{1}{2}) \neq 0$ then $\sigma(P, c) = \{0\} \cup \sigma(P_i, c)$.

In particular, if $P = P_1^2$ then the identity P is degenerated.

Grafting

Given an algebra identity P, one frequently use its 'derivatives', i.e. an identity P_1 (normally of a higher order) obtained from the linearization of the original identity. In this case, any algebra satisfying $P = 0$ also satisfies $P_1 = 0$. We briefly consider the simplest variant of this construction below.

Given a nonassociative polynomial, one can substitute another monomial (grafting) as follows:

$$x^3 \to D(x^3; y) = 2x \bullet (x \bullet y) + x^2 \bullet y$$

$$\text{replace } y \text{ by } x^2$$

$$\to 2x \bullet (x \bullet x^2) + x^2 \bullet x^2 = 2x^4 + x^2 \bullet x^2,$$

while

$$x^2 \to D(x^2; y) = 2x \bullet y \quad \to \quad \text{replace } y \text{ by } x^3 \quad \to \quad 2x \bullet x^2.$$

We denote this by $D(x^3; x^2)$, or in general by $D(P; Q)$.

Theorem 4

$$\rho(D(P; Q), q) = \rho(P, q) \cdot (\rho(Q, q) + (\rho(P, 1) - 1)\rho(Q, 1)).$$

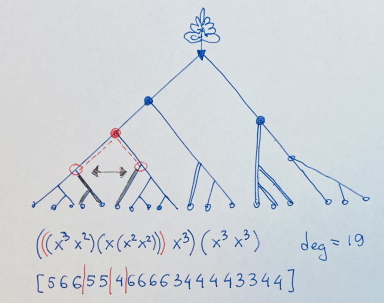
In particular,

- *The Peirce spectrum increases w.r.t. grafting.*
- *A grafting of a strange polynomial is a strange polynomial again.*

Definition

Two monomials x^α and x^β are **isomeric** ($x^\alpha \simeq x^\beta$) if any of the following equivalent conditions satisfied:

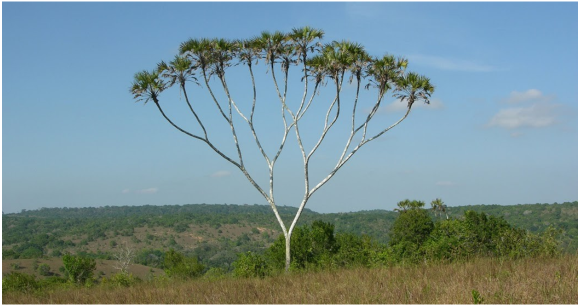
- $\rho(x^\alpha) = \rho(x^\beta)$.
- The unordered path-length sequences of x^α and x^β coincide.



Some relevant questions

- How to characterize two isomeric monomials?
- How to classify algebras satisfying 'strange' identities, i.e. $x^\alpha - x^\beta = 0$ for some $\rho(x^\alpha) = \rho(x^\beta)$.
- What is about the simplest nontrivial case $R := (x^3)^2 - (x^2)^3 = 0$?
- Do there exist any algebras satisfying this identity distinct from medial, power-associative or three-nilpotent ones?

THANK YOU FOR YOUR ATTENTION!



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