News on the Graded Classification Conjecture for Leavitt Path Algebras

joint work with R. Hazrat, A. Sebandal

Wolfgang Bock

Linnæus Universitetet, Växjö, Sweden

SNAG, BTH, 2025

Linnæus University

Roozbeh Hazrat
Western Sydney University

Alfilgen Sebandal
Linnaeus University/
RCTP Jagna, PH

W. Bock, R. Hazrat, A. Sebandal

The graded classification conjectures hold for various finite representations of Leavitt path algebras

Journal of Algebra, Volume 672, 15 June 2025, Pages 303-333

Linnæus University

Directed Graph

E

Talented Monoid

T_E

Hazrat (2013)
Hazrat, Li (2020)

Leavitt Path Algebra

$L_K(E)$

Abrams, Pino (2005)
Ara, Moreno, Pardo (2007)

R. Hazrat (2013)

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

$\mathbb{Z}$-isomorphism of talented monoids

Graded equivalence of **categories of graded modules** over the Leavitt path algebra

Linnæus University

R. Hazrat (2013)

Graded Classification Conjecture

For finite graphs E and F:

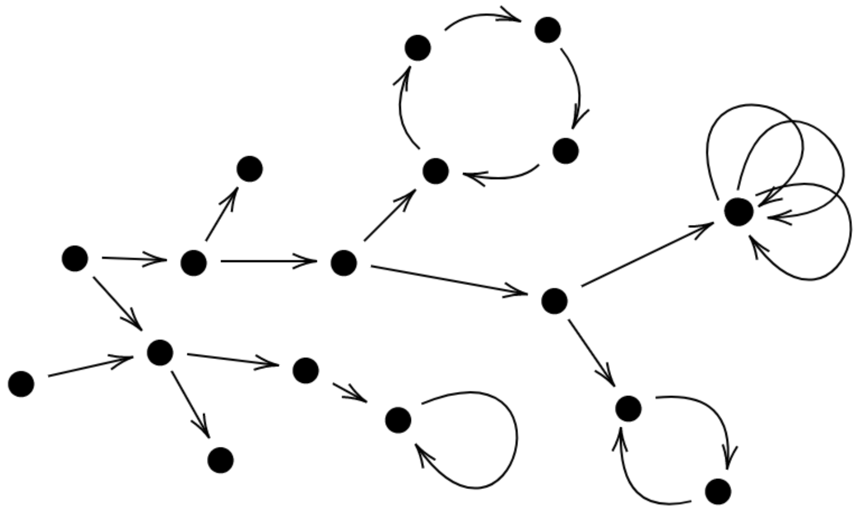
$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

✓ **Graded Classification Theorem:**

for Polycephaly Graphs

Directed Graph

E



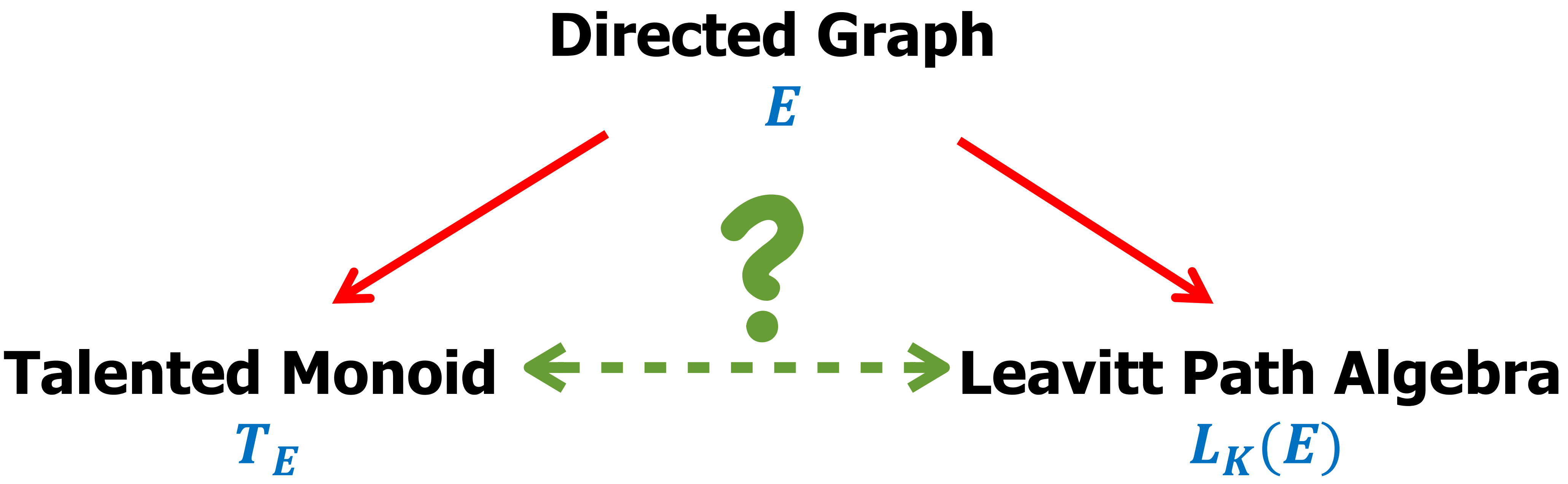
Polycephaly Graph

$E = (E^0, E^1, s, r)$

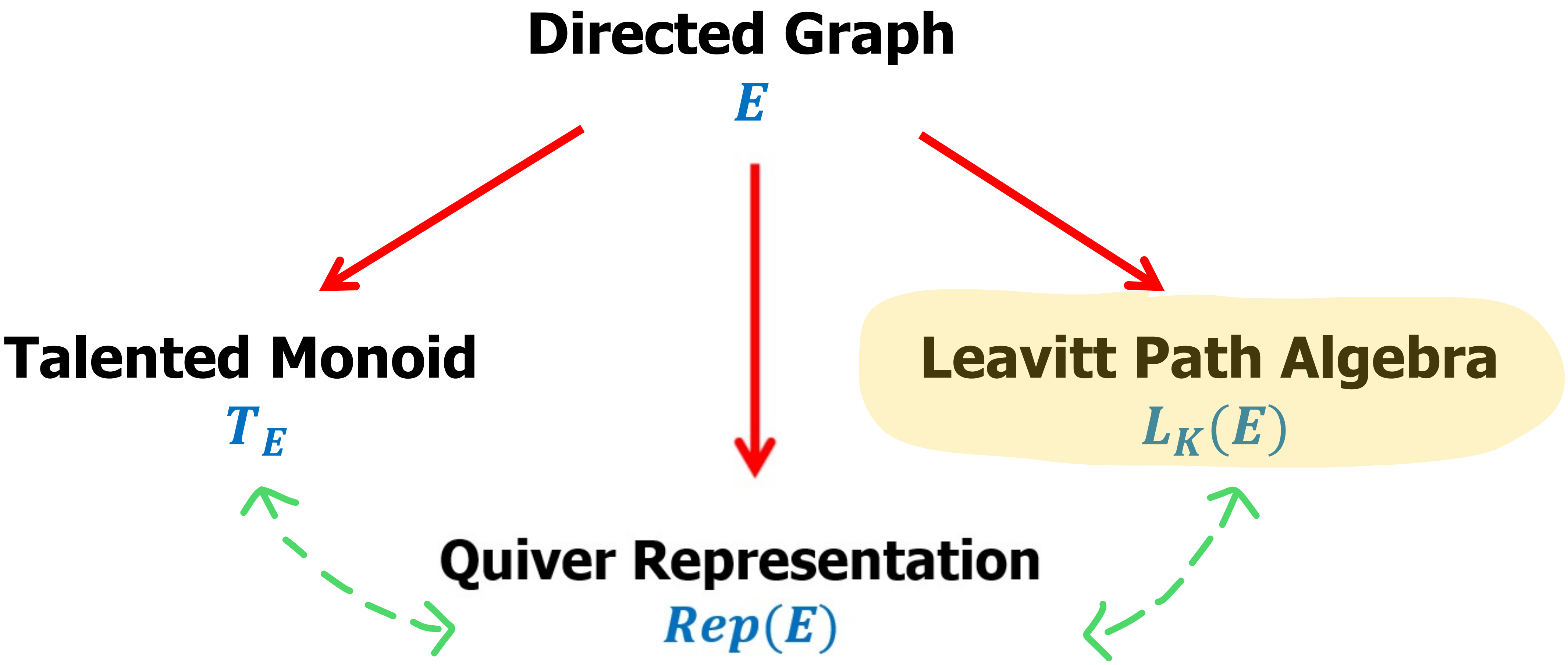
$s, r : E^1 \to E^0$

$s(e) \xrightarrow{e} r(e)$

source of e   range of e


Directed Graph
E
Talented Monoid
T_E
Leavitt Path Algebra
$L_K(E)$

Linnæus University



Path Algebra

For a graph E and a ring R with identity, the *Path algebra* of E, denoted by $P_K(E)$, is the R-algebra generated by the sets $\{v : v \in E^0\}$ and $\{e : e \in E^1\}$ with coefficients in R, subject to the relations:

(V) $v_i \cdot v_j = \delta_{i,j} v_i$ for every $v_i, v_j \in E^0$;

(E) $s(e) \cdot e = e = e \cdot r(e)$ for $e \in E^1$.

(V) $uu = u$

$uv = 0$

(E) $ue = e = ev$

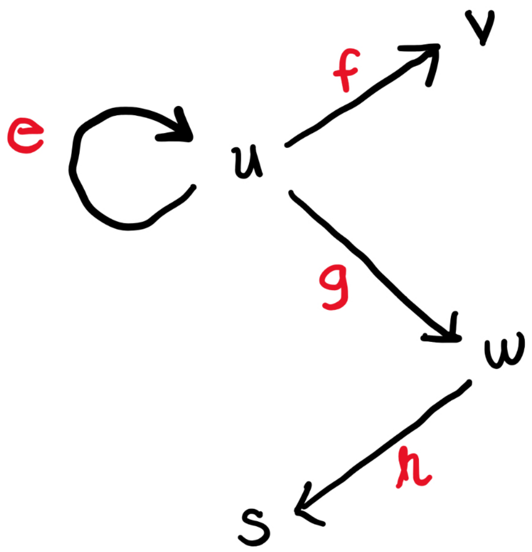
u = s(e), v = r(e)

Linnæus University

Another way of looking at it...

$$P_K(E) = \frac{K\langle E^0 \cup E^1 \rangle}{\langle (V), (E) \rangle}$$

E :



In $K\langle E^0 \cup E^1 \rangle$:

$$vv + vw + uf + fg$$

In $P_K(E)$:

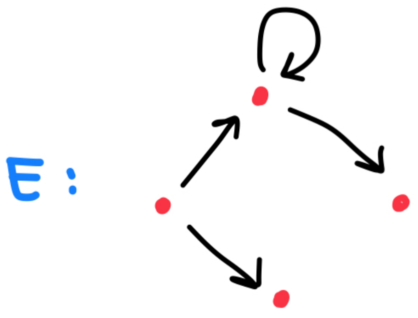
$$vv + vw + uf + gf$$

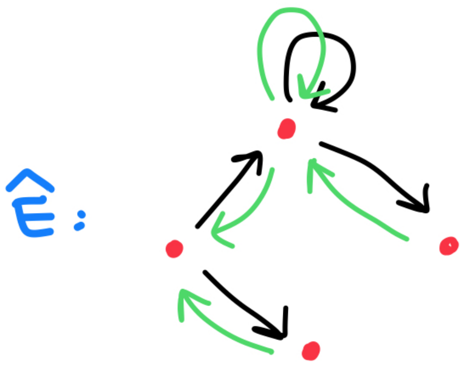
$$vv \to v, \quad vw \to 0, \quad uf \to f, \quad gf \to (gw)(uf) = g(wu)f = g0f = 0$$

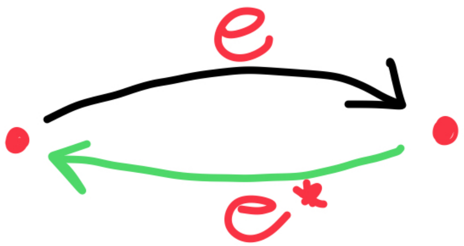
$$= v + f$$

Leavitt Path Algebra

For a graph E and a ring R with identity, the *Leavitt path algebra* of E, denoted by $L_R(E)$, is the path algebra over the double graph $\hat{E}$ with additional relations







Leavitt Path Algebra

For a graph E and a ring R with identity, the *Leavitt path algebra* of E, denoted by $L_R(E)$, is the path algebra over the double graph $\hat{E}$ with additional relations

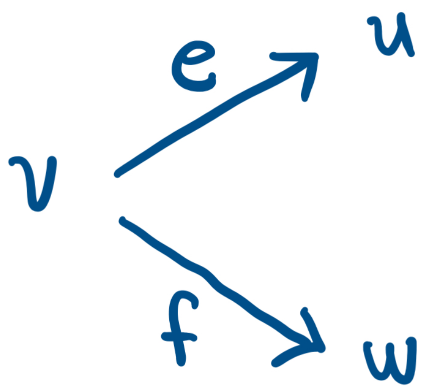
(CK1) $e^*f = \delta_{e,f}r(e)$ for all $e,f \in E^1$;

(CK2) $v = \sum_{e \in s^{-1}(v)} ee^*$ for every $v \in Reg(E)$.

$L_K(E) = P_K(\hat{E}) / \langle CK1, CK2 \rangle$

Linnæus University

Example



(CK1)

$$e^*e = r(e) = u$$

$$f^*f = r(f) = w$$

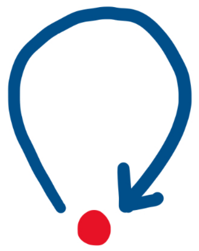
$$e^*f = 0 = f^*e$$

(CK2)

$$v = ee^* + ff^*$$

Fundamental Examples

E:



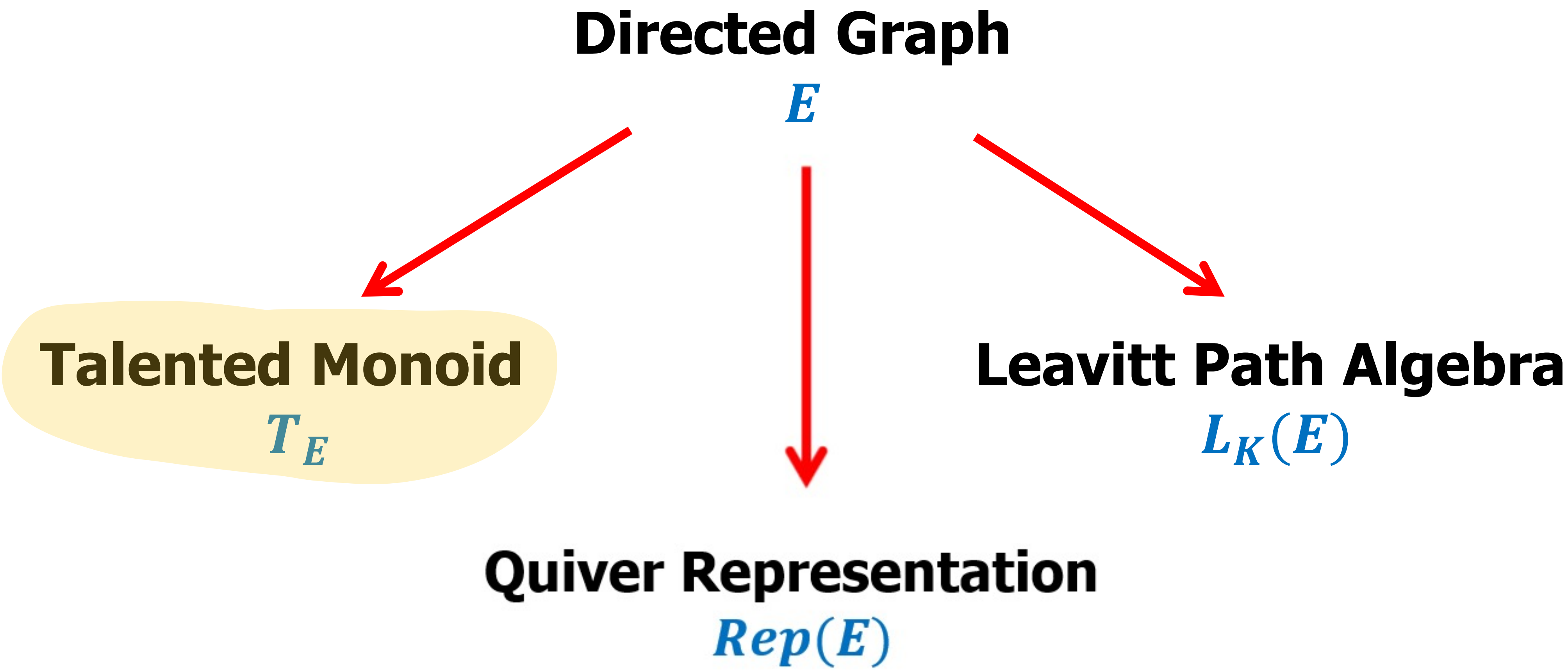
F:

$$P_K(E) = K[x]$$

$$P_K(E) = \text{upper triangular} M_n(K)$$

$$L_K(E) = K[x, x^{-1}]$$

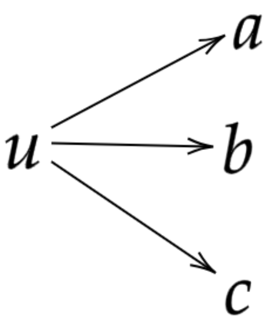
$$L_K(F) = M_n(K)$$


Directed Graph
E
Talented Monoid
T_E
Leavitt Path Algebra
$L_K(E)$
Quiver Representation
$Rep(E)$

Let E be a row-finite graph. The *talented monoid* of E, denoted by T_E, is the abelian monoid generated by $\{v(i) : v \in E^0, i \in \mathbb{Z}\}$, subject to

$$v(i) = \sum_{e \in s^{-1}(v)} r(e)(i+1)$$

for every $i \in \mathbb{Z}$ and every $v \in E^0$ that is not a sink.



$$\begin{aligned} u(1) &= a(1+1) + b(1+1) + c(1+1) \\ &= a(2) + b(2) + c(2) \end{aligned}$$

Let E be a row-finite graph. The *talented monoid* of E, denoted by T_E, is the abelian monoid generated by $\{v(i) : v \in E^0, i \in \mathbb{Z}\}$, subject to

$$v(i) = \sum_{e \in s^{-1}(v)} r(e)(i+1)$$

for every $i \in \mathbb{Z}$ and every $v \in E^0$ that is not a sink.

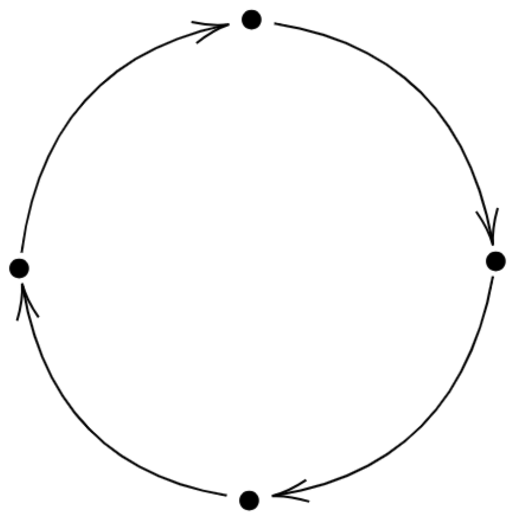
Ara, Hazrat, Li, Sims (2018)

$\boldsymbol{T_E}$ - monoid of graded finitely generated projective modules over $L_K(E)$

Linnæus University

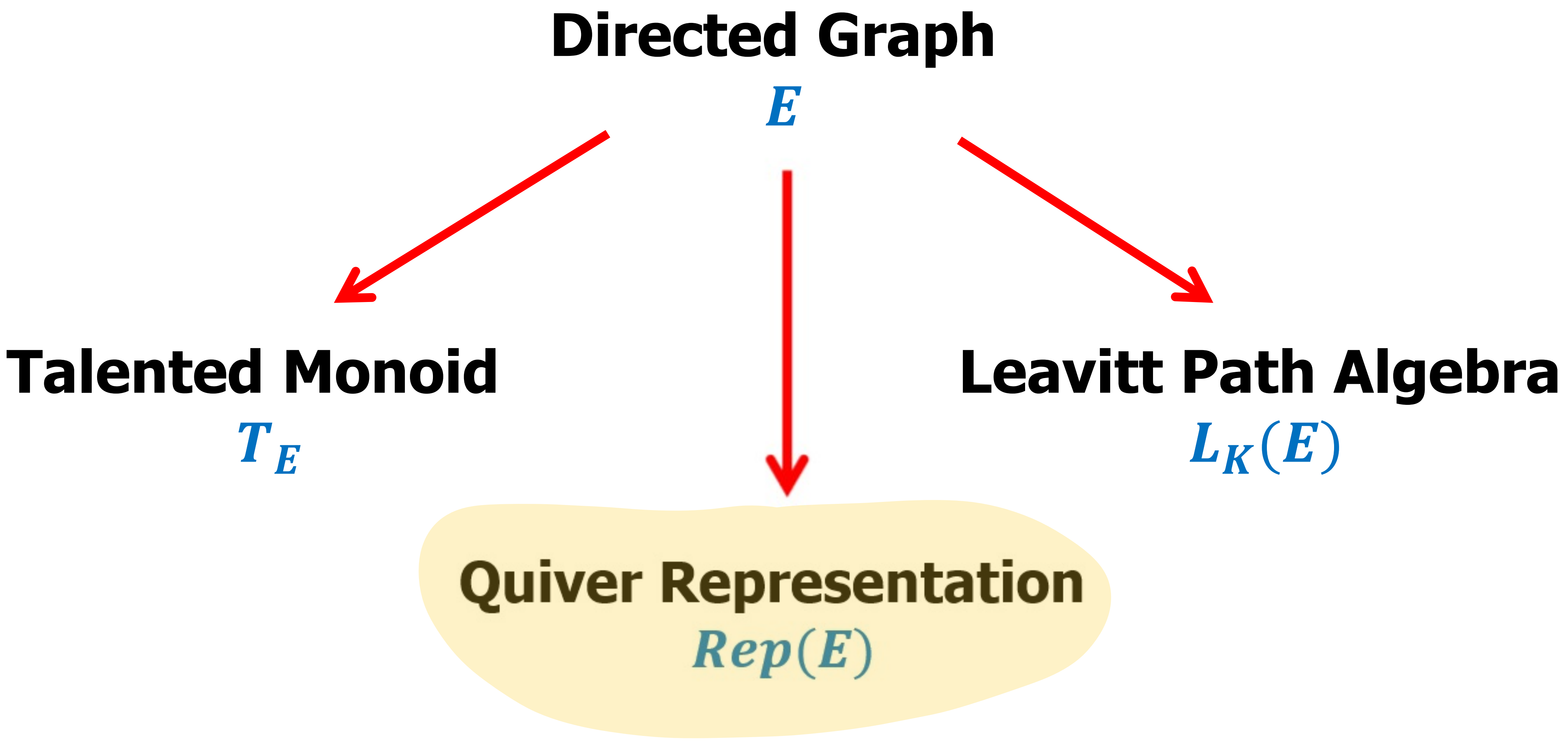
Example

E:



$$T_E = \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}$$

Linnæus University

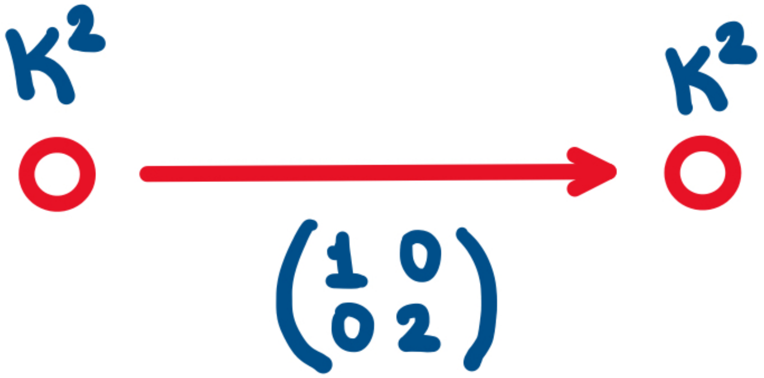


Linnæus University

Quiver Representation

E - directed graph

- putting vector spaces on vertices and linear transformations on arrows



Linnæus University

A **representation** of E is a functor of categories

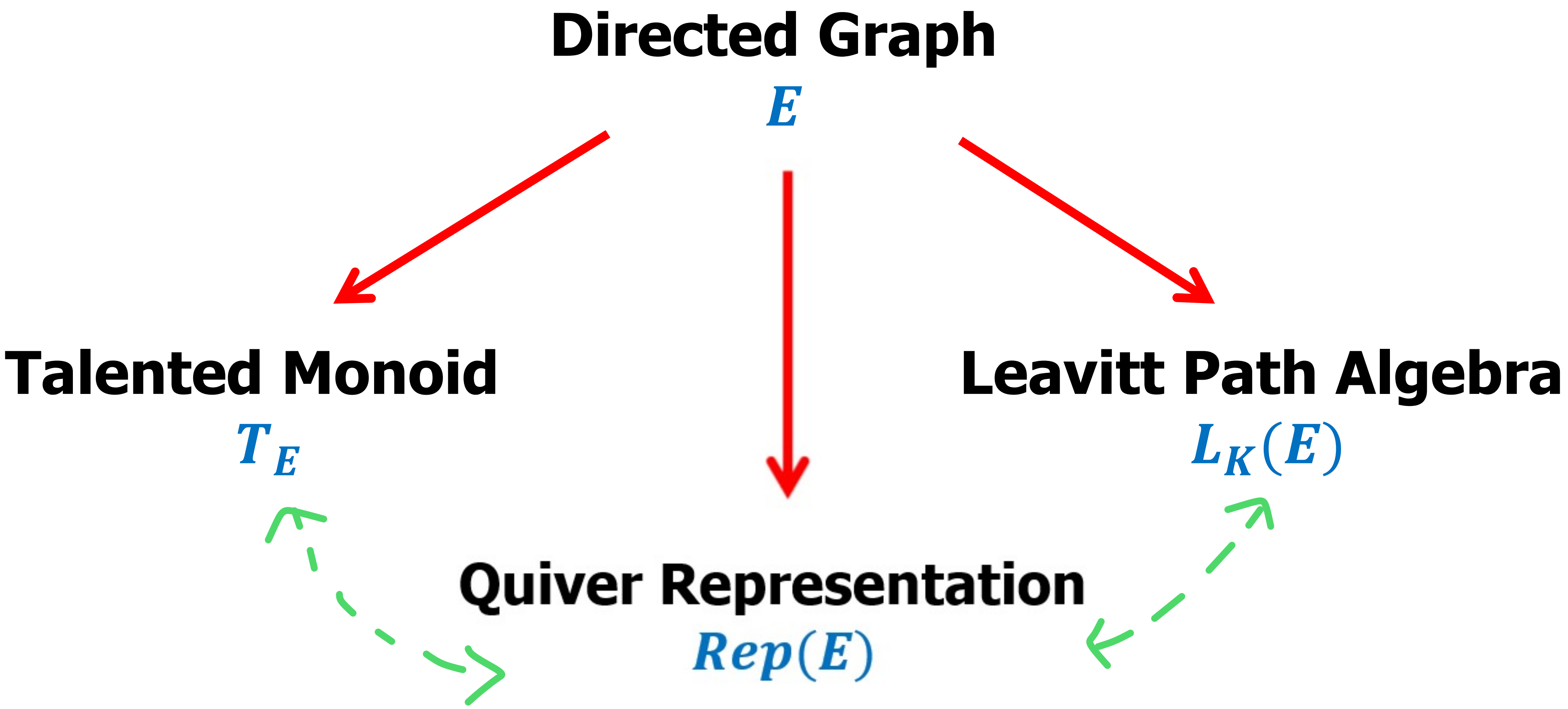
$$\rho : E \longrightarrow \mathrm{Vec}\, K$$

$$v \longmapsto \rho(v)$$ - vector space over K

$$u \xrightarrow{e} v \longmapsto \rho(u) \xrightarrow{\rho(e)} \rho(v)$$ - linear transformation

$\mathrm{Rep}(E)$ - category of representations of E

Linnæus University



R. Hazrat (2013)

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \quad \Longleftrightarrow \quad Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

$\mathbb{Z}$-isomorphism of talented monoids

Graded equivalence of **categories of graded modules** over the Leavitt path algebra

Linnæus University

As categories

$$\mathrm{Rep}(E) \sim \text{Mod-}P_K(E)$$

- category of modules over $P_K(E)$

Representation $\longmapsto$ **$P_K(E)$-module**

$$\rho : E \to \mathrm{Vec}\,K$$

$$v \mapsto \rho(v)$$

$$e : u \to v \mapsto \rho(e) : \rho(u) \to \rho(v)$$

$$M_\rho = \bigoplus_{v \in E^0} \rho(v)$$

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

As categories

$$\mathrm{Rep}(E) \sim \mathrm{Mod}\text{-}P_K(E)$$

- category of modules over $P_K(E)$

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

Graph E → Covering Graph $\bar{E}$

$E = (E^0, E^1, r, s)$

$\bar{E} = (\bar{E}^0, \bar{E}^1, \bar{r}, \bar{s})$

$\bar{E}^0 = \{v_i : v \in E^0, i \in \mathbb{Z}\}$ $\quad \bar{s}(e_i) = s(e)_i$

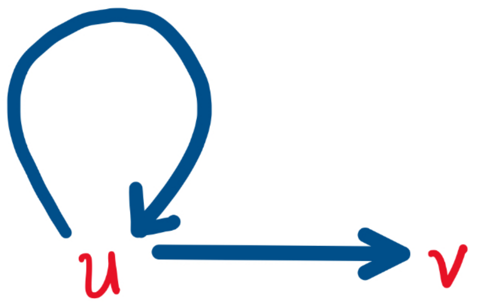
$\bar{E}^1 = \{e_i : e \in E^1, i \in \mathbb{Z}\}$ $\quad \bar{r}(e_i) = r(e)_{i+1}$

Linnæus University

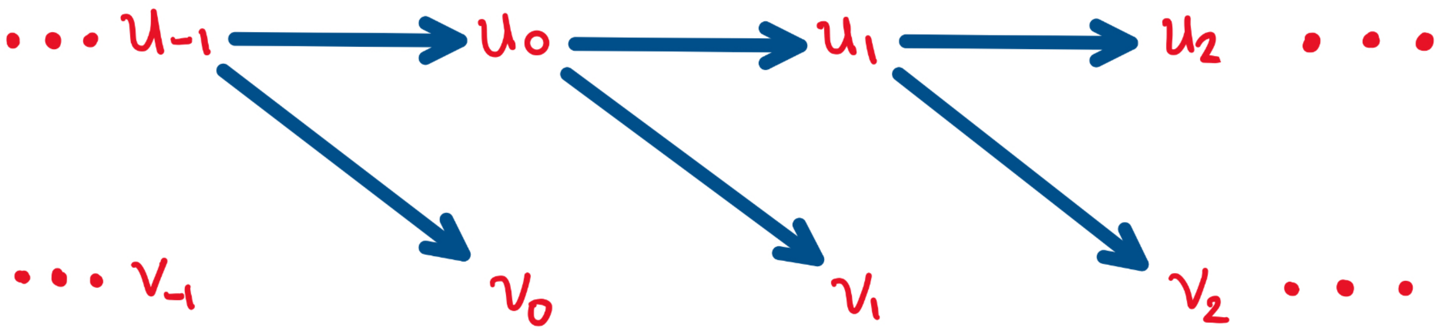
Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$



E



$\overline{E}$

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

$\bar{E}$ - covering graph

$$\mathrm{Rep}(\bar{E}) \sim \mathrm{Mod}\text{-}P_K(\bar{E}) = Gr\text{-}P_K(E)$$

- category of graded modules over $P_K(E)$

Recall:

$$L_k(E) = P_k(\hat{E}) / \langle c_k \rangle$$

$$\mathrm{Rep}(\bar{E}) \sim \mathrm{Gr}\text{-}P_k(E)$$

$$\mathrm{Rep}(\bar{\hat{E}}, c_k) \sim \mathrm{Gr}\text{-}P_k(\hat{E}) / \langle c_k \rangle$$

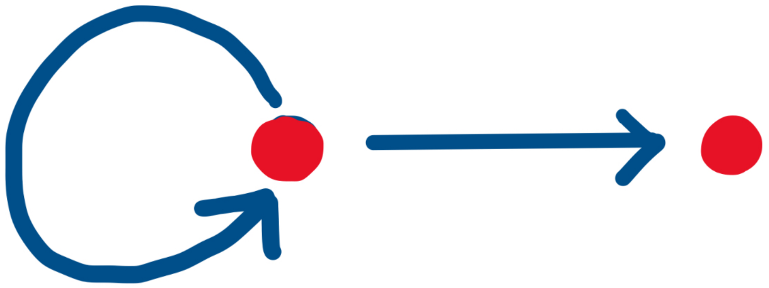
Linnæus University

Recall:

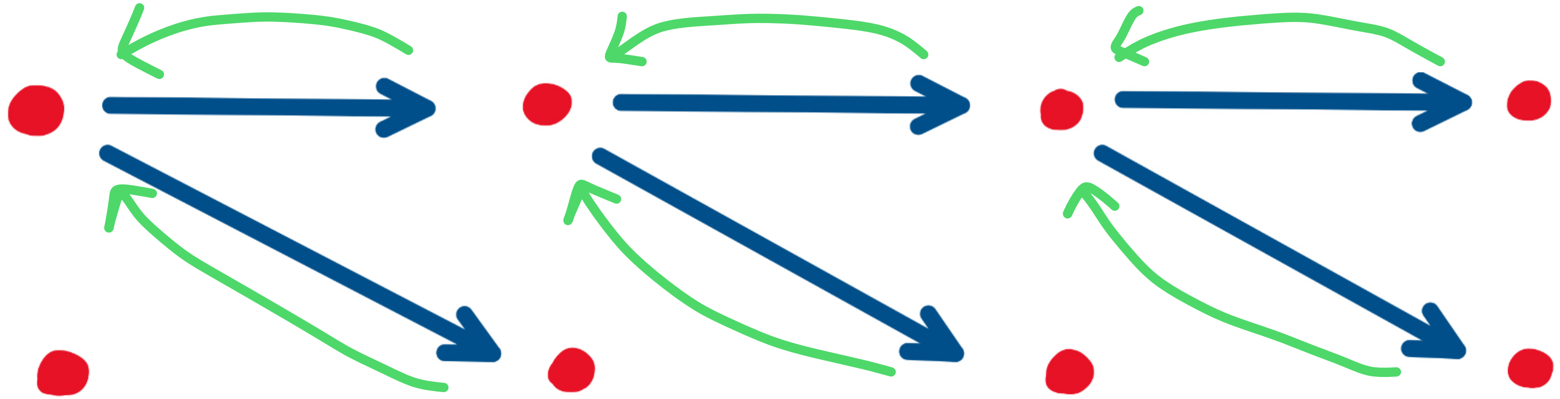
$$L_k(E) = P_k(\hat{E}) / \langle c_k \rangle$$

$$\mathrm{Rep}(\bar{\hat{E}}, c_k) \sim \mathrm{Gr}\text{-}L_k(E)$$

E :

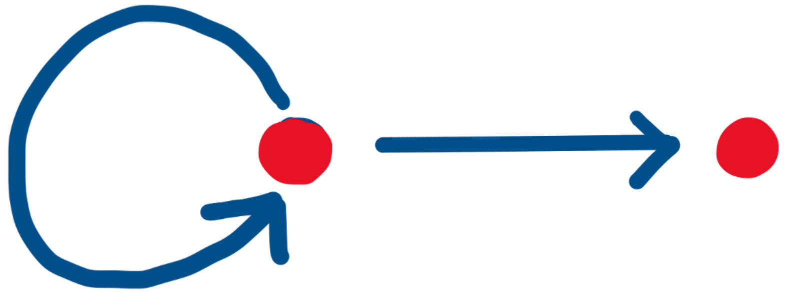


$\bar{\hat{E}}$:

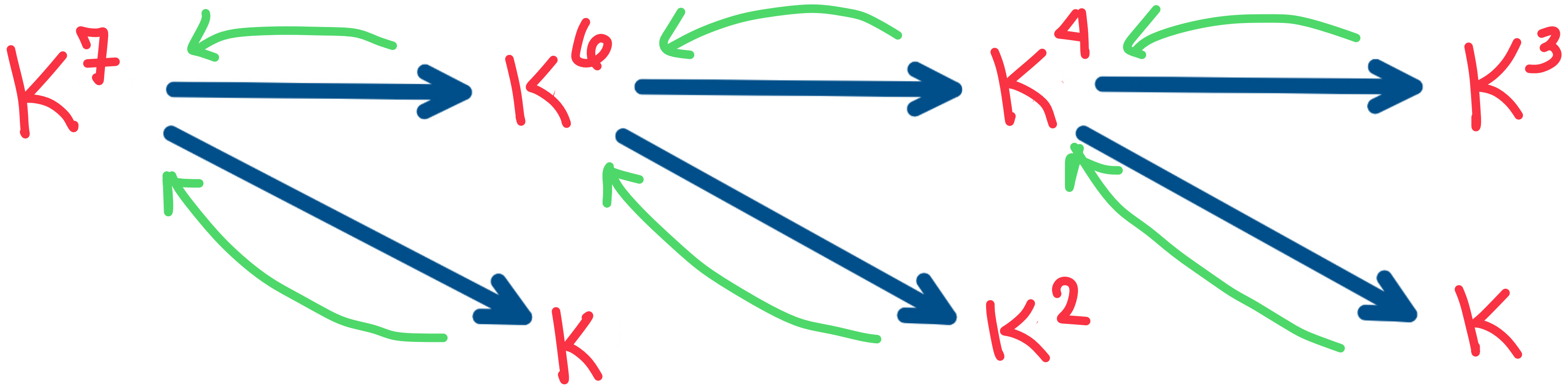


Linnæus University

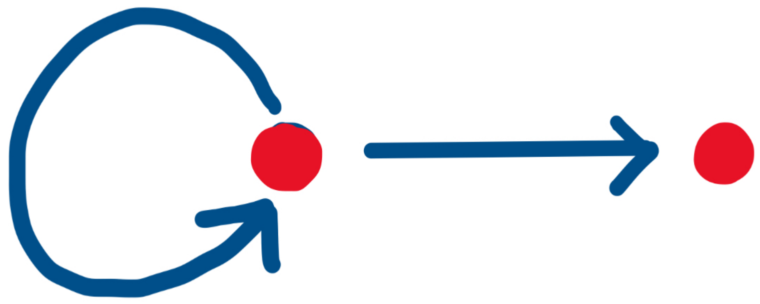
E :



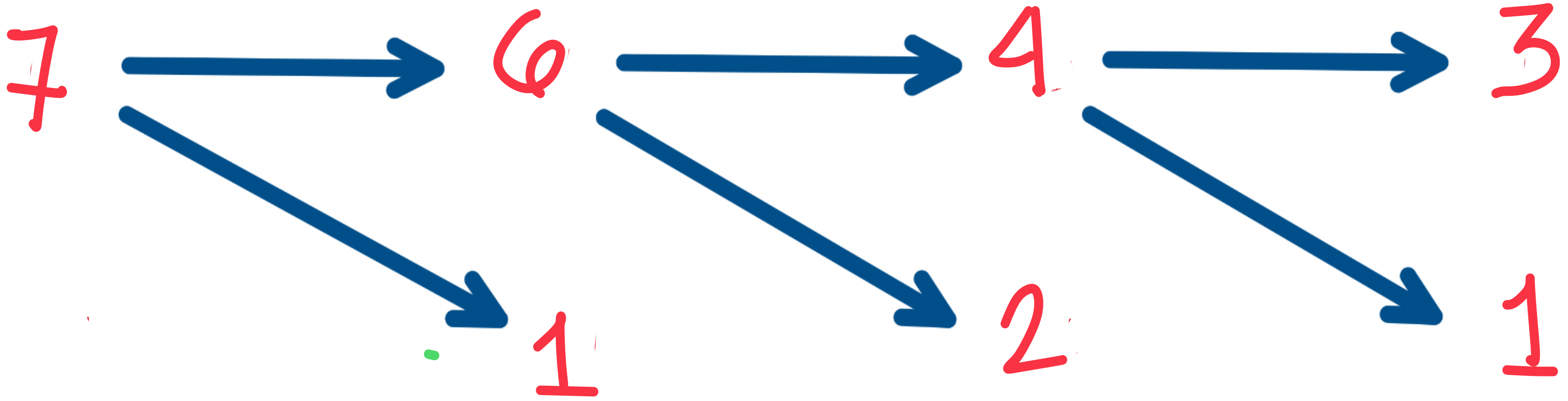
$\mathrm{Rep}(\hat{E}, CK)$



E :



$\mathrm{Rep}(\bar{\hat{E}}, CK)$



Linnæus University

Theorem (Koc, Ozaydin 2018)

If E is a row-finite graph, then the category of unital $L_K(E)$-module is equivalent to the full subcategory of representation φ of $\hat{E}$ satisfying:

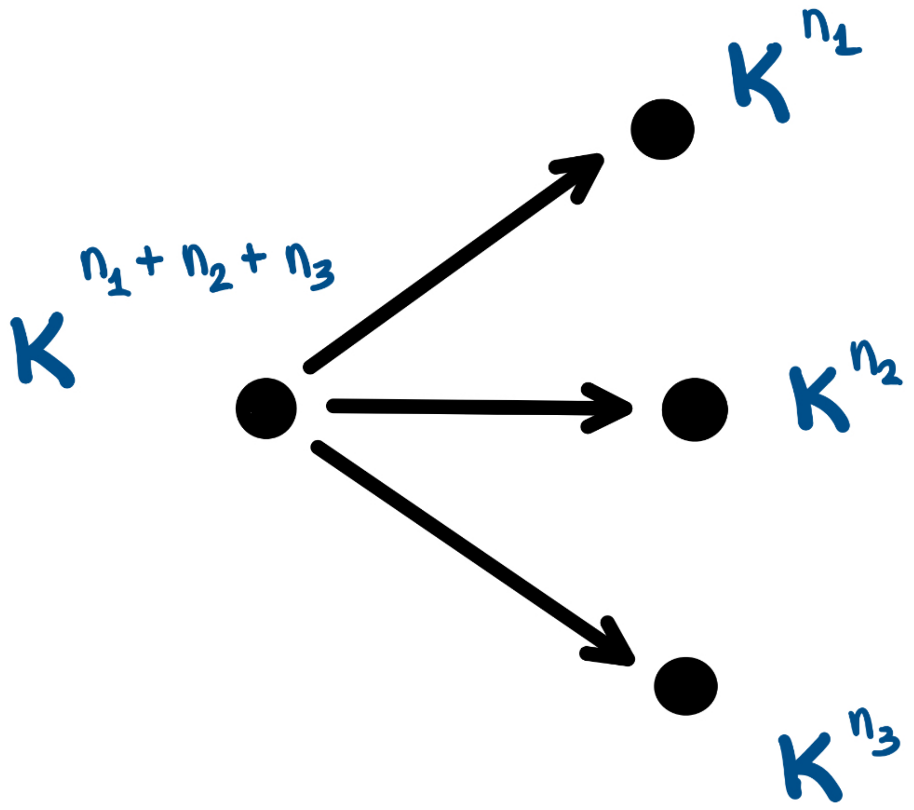
$$\varphi(e)_{s(e)=v} : \varphi(v) \longrightarrow \bigoplus_{s(e)=v} \varphi(r(e))$$

is an isomorphism.

$$\varphi(v) \cong \bigoplus \varphi(r(e))$$

$$\Downarrow$$

$$\dim(\varphi(v)) = \sum \dim(\varphi(r(e)))$$



How to use this ?

Linnæus University

$$\mathrm{Rep}(\overline{\hat{E}}, CK) \sim \mathrm{Gr}\text{-}L_K(E)$$

How about the conjecture ?

$$\begin{array}{ccccc} T_E \cong T_F & \Longleftrightarrow & \mathrm{Gr}\text{-}L_K(E) & \approx_{gr} & \mathrm{Gr}\text{-}L_K(F) \\ & \searrow\!\!\!\nwarrow & \Updownarrow & & \Updownarrow \\ & & \mathrm{Rep}(\overline{\hat{E}}, CK) & \sim & \mathrm{Rep}(\overline{\hat{F}}, CK) \end{array}$$

↳ should commute with the shift functor

Back Track a bit ...

Motivation:

Koc and Ozaydin (2018, 2020)

- characterized finite-dimensional representations of LPA thru maximal cycles and sinks.

Linnæus University

Theorem

Let E and F be finite graphs such that

$$T_E \cong T_F.$$

Then there is a one-to-one correspondence between the maximal sinks and maximal cycles in E and F with the same length.

Linnæus University

Proporition

Let E be a finite graph such that sources are isolated vertices. Then

$$\rho : \hat{\bar{E}} \to \mathrm{Vec}\,k \quad \text{in} \quad \mathrm{rep}(\hat{E}, r_{cu})$$

has the form

$$\rho(v) = \begin{cases} k^{n_c} & \text{if } v \in C \text{ a maximal cycle} \\ k^{n_v} & \text{if } v \text{ is an isolated vertex} \\ 0 & \text{otherwise} \end{cases}$$

Graded Classification Conjecture

For finite graphs E and F:

$$T_E \cong T_F \iff Gr\text{-}L_K(E) \approx_{gr} Gr\text{-}L_K(F)$$

The Graded Classification is true for finite-dimensional case!

☑ $\dim\left(\bigoplus_{i\in\mathbb{Z}} M_i\right) < \infty$

☑ Sinkless graphs:

$\dim(M_i) < \infty$ for each i

Theorem

Let E and F be finite graphs with no sources and sinks and let $L_{\mathsf{k}}(E)$ and $L_{\mathsf{k}}(F)$ be their associated Leavitt path algebras with coefficients in a field k, respectively. If there is an order-preserving $\mathbb{Z}[x, x^{-1}]$-module isomorphism $K_0^{\mathrm{gr}}(L_{\mathsf{k}}(E)) \cong K_0^{\mathrm{gr}}(L_{\mathsf{k}}(F))$, then there is an equivalence,

$$\text{mod-}\, L_{\mathsf{k}}(E) \approx \text{mod-}\, L_{\mathsf{k}}(F)$$

as well as graded equivalences

$$\text{gr-}L_{\mathsf{k}}(E) \approx_{\mathrm{gr}} \text{gr-}L_{\mathsf{k}}(F) \textit{ and } \mathrm{gr}^{\mathbb{Z}}\text{-}L_{\mathsf{k}}(E) \approx_{\mathrm{gr}} \mathrm{gr}^{\mathbb{Z}}\text{-}L_{\mathsf{k}}(F).$$

Thank you very much!

Tack så mycket!

Linnæus University