On (n-)representation-finite self-injective and hereditary algebras

SNAG meeting 2026-5-29, Linköping

Basic problem: Understand (finitely generated) modules of finite-dimensional *k*-algebras.

Problem with the problem: Most algebras are wild.

Representation type

gldim	Rep. fin.	tame	wild
0	✓		
1	ADE	$\tilde{A}\ \tilde{D}\ \tilde{E}$	...
2	?	?	?
$\vdots$	?	?	?
∞	?	?	?

Throughout: k: field, A: fin.-dim. k-algebra.
A-modules are finitely generated ($\Leftrightarrow$ finite dimensional)

A is:

- **hereditary** if $\operatorname{gldim} A \leq 1$ ($\Leftrightarrow$ submodules of projective modules are projective)
- **self-injective** if A_A is injective ($\Leftrightarrow \operatorname{proj} A = \operatorname{inj} A$) $\operatorname{gldim} A = \infty$ or 0

If $k = \bar{k}$: $\operatorname{mod} A \simeq \operatorname{mod}(kQ/I)$

$$\begin{cases} Q: \text{finite quiver} \\ kQ: \text{path algebra} \\ I \triangleleft kQ \text{ admissible ideal} \end{cases}$$

$\left[R^n \subset I \subset R^2 \right.$ (arrow ideal) $\left. \right]$

A hereditary $\Longleftrightarrow I = 0$.

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Auslander–Reiten quiver Γ_A:

vertices: $[M]$, $M \in \operatorname{mod} A$ indecomposable

arrows: $[M] \longrightarrow [N]$ irreducible morphisms $M \to N$

$$\#([M] \to [N]) = \dim_k \left(\frac{\operatorname{rad}(M,N)}{\operatorname{rad}^2(M,N)} \right)$$

Ex: $A = kQ$, $Q = (1 \xrightarrow{a} 2 \xrightarrow{b} 3)$

$$A \simeq \begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

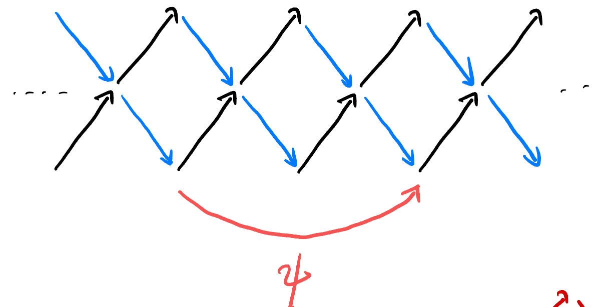
$$\begin{tikzcd}
 & & P_1 = I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & I_2 \arrow[dr] & \\
S_3 = P_3 \arrow[ur] \arrow[rr, dashed, no head] & & S_2 \arrow[ur] \arrow[rr, dashed, no head] & & I_1 = S_1
\end{tikzcd}$$

Gabriel's thm: ($k=\bar{k}$) kQ is representation finite (RF) $\iff$ Q is Dynkin of type A, D or E.

Riedtmann ('80-83): (building on Gabriel's covering theory)

Q : Dynkin (ADE)

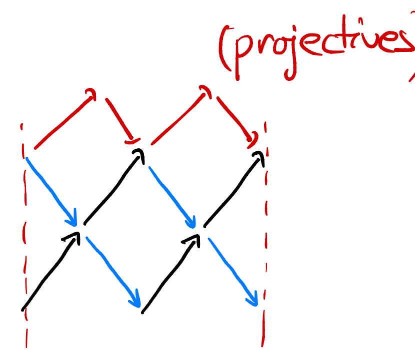
The repetition quiver $\mathbb{Z}Q$:



$\psi : \mathbb{Z}Q \to \mathbb{Z}Q$ automorphism acting freely on vertices.

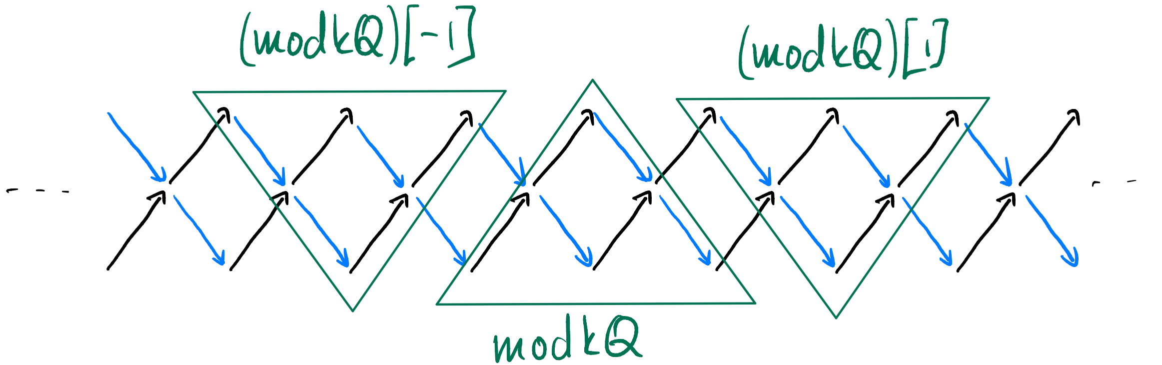
Then $\mathbb{Z}Q/\psi$ is the stable AR quiver of a RF self-injective algebra.

$\mathbb{Z}Q/\psi$:



Modern point of view:

$\mathbb{Z}Q$ is the AR quiver of $\mathcal{D}^b(kQ)$ (many copies of $\operatorname{mod} kQ$)



Let A be such that $\mathcal{D}^b(A) \simeq \mathcal{D}^b(kQ)$ (Q : Dynkin)

$$\hat{A} = \begin{pmatrix} \ddots & & & & \\ & A & & & \\ & DA & A & & \\ & & DA & \ddots & \\ & & & A & \\ & & & DA & \ddots \\ & & & & \ddots \end{pmatrix}$$

($\sim \operatorname{proj}^{\mathbb{Z}} T(kQ)$) repetitive algebra

$\varphi : \hat{A} \to \hat{A}$ admissible automorphism $\dim_k (\hat{A}/\varphi) < \infty$

Then: 1) The orbit algebra $\hat{A}/\varphi$ is RF self-injective.

2) If $k = \bar{k}$ and $\operatorname{char} k \neq 2$ then every RF self-injective algebra arises in this way.

(up to Morita equivalence)

The AR quiver of $\hat{A}/\varphi$:

- Happel's thm: If $\operatorname{gldim} A < \infty$ then $\mathcal{D}^b(A) \simeq \underline{\mathrm{mod}}\,\hat{A}$

  $\underline{\mathrm{mod}}\,\hat{A} = \dfrac{\mathrm{mod}\,\hat{A}}{(\text{proj.})}$ stable module cat.

- Covering functor: $F : \hat{A} \to \hat{A}/\varphi$

  $\rightsquigarrow$ push-down functor $F_\lambda : \underline{\mathrm{mod}}\,\hat{A} \longrightarrow \underline{\mathrm{mod}}\,(\hat{A}/\varphi)$

$$
\begin{tikzcd}
\mathcal{D}^b(kQ) \simeq \mathcal{D}^b(A) \arrow[r, "\simeq"] & \underline{\mathrm{mod}}\,\hat{A} \arrow[d, "F_\lambda"', "\substack{\text{faithful}\\ \text{dense}}"] \\
 & \underline{\mathrm{mod}}(\hat{A}/\varphi) \simeq (\underline{\mathrm{mod}}\,\hat{A})/\varphi_*
\end{tikzcd}
$$

induced by φ

$\rightsquigarrow \Gamma_{\hat{A}/\varphi} \simeq \mathbb{Z}Q/\varphi_*$

Higher-dimensional AR theory

Wild algebras through subcat.
$\mathcal{M} \subset \operatorname{mod} A$ with good
homological properties.

Def: $n \geq 1$. 1) $M \in \operatorname{mod} A$ is **n-cluster tilting** (CT) if

$$\operatorname{add} M = \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i(M, X) = 0 \ \forall i = 1, \dots, n-1\}$$
$$= \{X \in \operatorname{mod} A \mid \operatorname{Ext}^i(X, M) = 0 \ \forall i = 1, \dots, n-1\}.$$

2) A is n-representation-finite if it has an n-CT module.

$\operatorname{add} M$ has a "higher-dimensional" AR theory,
analogous to that of a RF algebra.

Want a higher-dim version of Riedtmann's result.

n-RF of global dimension n corr. to RF hereditary

Let $A \underset{\text{der}}{\sim} A'$, A' n-RF, $\operatorname{gldim} A' = n$.

$M \subset \operatorname{mod} A'$ n-CT $\rightsquigarrow \mathcal{U} = \operatorname{add}\{M[n\ell] \mid \ell \in \mathbb{Z}\} \subset \mathcal{D}^b(A')$

n-CT subcategory (corr. to $\mathcal{D}^b(kQ)$)

$\varphi : \hat{A} \to \hat{A}$ admissible autom.

such that $\varphi_*(\tilde{\mathcal{U}}) \subset \tilde{\mathcal{U}}$.

$$\begin{tikzcd}
\mathcal{D}^b(A') \arrow[r, phantom, "\simeq"] & \mathcal{D}^b(A) \arrow[r, "\sim"] & \underline{\operatorname{mod}}(\hat{A}) \arrow[dd, "F_\lambda"] \\
\mathcal{U} \arrow[u, phantom, "\cup"] \arrow[rr, "\sim"] & & \tilde{\mathcal{U}} \arrow[u, phantom, "\cup"] \arrow[d] \\
& & F_\lambda(\tilde{\mathcal{U}}) \subset \underline{\operatorname{mod}}(\hat{A}/\varphi)
\end{tikzcd}$$

Then:
$$\boxed{\begin{array}{l} \hat{A}/\varphi \text{ is } n\text{-RF} \\ F_\lambda(\tilde{\mathcal{U}}) \subset \underline{\operatorname{mod}}(\hat{A}/\varphi) \text{ is } n\text{-CT} \end{array}}$$

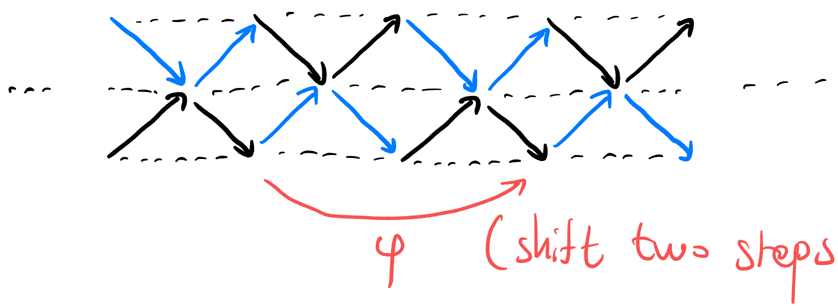
Ex: $A = k(1 \xrightarrow{a} 2 \xrightarrow{b} 3)/\langle ab \rangle$

Γ_A :

$\operatorname{gldim} A = 2$

$M = (001) \oplus (011) \oplus (110) \oplus (001)$ is 2-CT

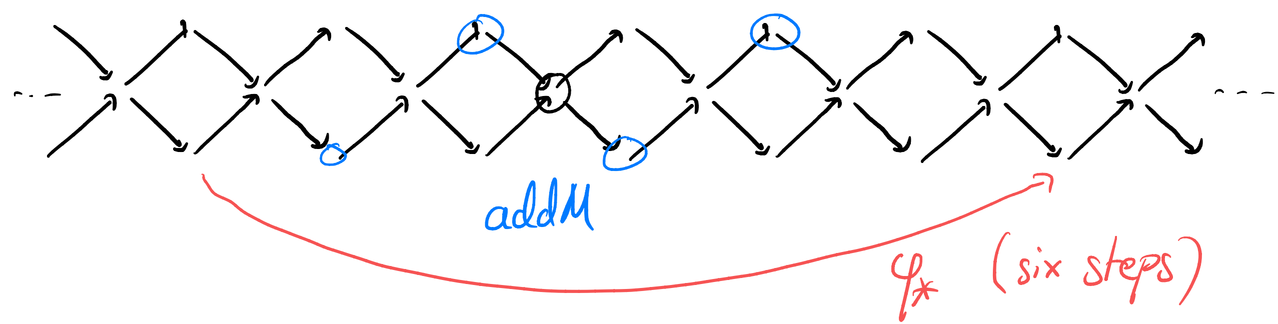
$\hat{A}$:



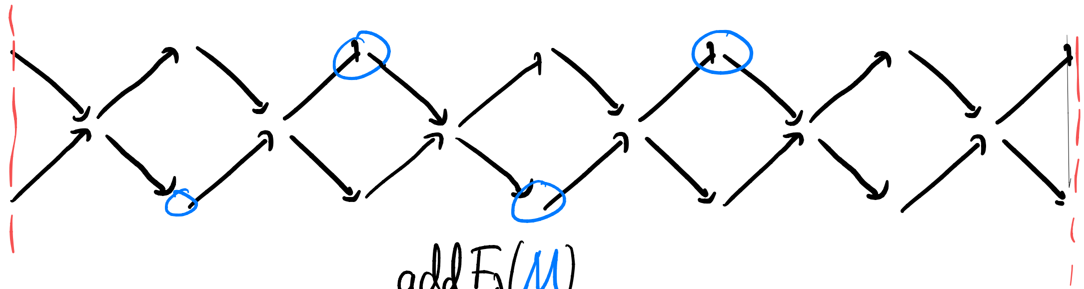
$\mathcal{D}^b(A)$:

$\cong$

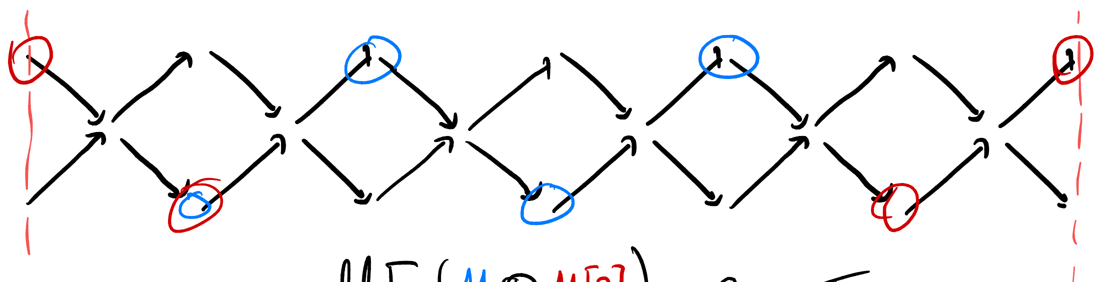
$\underline{\mathrm{mod}}\, \hat{A}$



$$\underline{\mathrm{mod}}(\hat{A}/\varphi) \simeq \mathcal{D}^b(A)/\varphi_*$$



$\mathrm{add}\, F_\lambda(M)$



$\mathrm{add}\, F_\lambda(M \oplus M[2])$ 2-CT

$\mathrm{mod}(\hat{A}/\varphi)$: