

Factor systems of strongly graded rings

Joakim Arnlind
Linköping University

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References

This is joint work with Stefan Wagner, and based on

[Factor systems and geometric structures of strongly graded rings](#)

J. A. and Stefan Wagner, [arXiv:2605.08919](#)

Before introducing you to our work, let us fix some notation concerning graded rings.

Graded rings

If G is a group, then a ring S is called G -graded if it admits a decomposition

$$S = \bigoplus_{g \in G} S_g$$

into additive subgroups $(S_g)_{g \in G}$, called the *homogeneous components*, such that

$$S_g S_h \subseteq S_{gh} \quad \text{for all } g, h \in G.$$

The component S_e corresponding to the identity element $e \in G$ is a subring of S , called the *principal component* (and will often be denoted by R in this talk).

Furthermore, if $S_g S_h = S_{gh}$ then S is called *strongly graded*.

Homogeneous components as bimodules

The homogeneous components S_g are bimodules over the principal component $R = S_e$, due to the grading requirement.

If S is strongly graded then S_g is an invertible R -bimodule with inverse given by $S_{g^{-1}}$; i.e

$$S_g \otimes_R S_{g^{-1}} \simeq R$$

Furthermore, S_g is projective as both a left and right R -module.

For an invertible R -bimodule P , every bimodule homomorphism is realized as left multiplication with an element of $Z(R)$, the center of R , i.e.

$$\text{Hom}_{R-R}(P) \simeq Z(R).$$

Introduction

- This project started out with trying to find natural classes of algebras and modules for which equivariant Levi-Civita connections could be constructed.
- In noncommutative geometry, strongly graded rings may be related to principal bundles where the principal component corresponds to functions on the base space.
- Via the Aityah sequence, we started to consider the problem of lifting derivations in graded rings, corresponding to lifting vector fields from the base space to the total space.
- This led us to "factor systems" and their usefulness in proving results and doing computations.
- Inspiration is taken from group representations, C^* dynamical systems and crossed products.

Crossed products

Let R be a ring. A pair consisting of a group homomorphism $\alpha : G \rightarrow \text{Aut}(R)$ and a map $\omega : G \times G \rightarrow R^\times$ is called a *factor system* if the following compatibility conditions are satisfied:

$$\begin{aligned}(\alpha_g \circ \alpha_h)(r) &= \omega(g, h)\alpha_{gh}(r)\omega(g, h)^{-1}, \\ \omega(g, h)\omega(gh, k) &= \alpha_g(\omega(h, k))\omega(g, hk)\end{aligned}$$

for all $g, h, k \in G$ and $r \in R$. Given such a factor system (α, ω) , the *crossed product* $R \rtimes_{(\alpha, \omega)} G$ is defined as the set of formal sums

$$\sum_{g \in G} r_g u_g, \quad r_g \in R,$$

with multiplication determined by

$$(r_g u_g)(r_h u_h) = r_g \alpha_g(r_h) \omega(g, h) u_{gh}.$$

In particular, $u_g r = \alpha_g(r) u_g$.

Crossed products

The crossed product $R \rtimes_{(\alpha, \omega)} G$ is a strongly G -graded ring

$$R \rtimes_{(\alpha, \omega)} G = \bigoplus_{g \in G} Ru_g$$

whose principal component is R and where each homogeneous component is a free module of rank one since u_g is invertible.

Even though crossed products, in the above sense, have a simple structure as strongly graded algebras, they still give rise to many interesting examples in mathematics.

However, for instance, strongly graded Leavitt path algebras is a class of strongly \mathbb{Z} -graded algebras that are in general not crossed products.

Factor systems

Can one describe an arbitrary strongly graded ring in a similar manner?
Starting from a ring R and group G together with an action of G on R ?

Let us start from the following formulation of a strong gradation:

Lemma (Dade '80)

A G -graded ring $S = \bigoplus_{g \in G} S_g$ is strongly G -graded if and only if for each $g \in G$ there exist a positive integer n_g and elements

$$x_g \in M_{n_g,1}(S_g) \quad \text{and} \quad y_g \in M_{n_g,1}(S_{g^{-1}})$$

such that $y_g^t x_g = 1$.

($M_{n,m}(A)$ denotes the set of $n \times m$ matrices with entries in a ring A , and we consider x_g and y_g as column vectors.)

Definition

Let $S = \bigoplus_{g \in G} S_g$ be a strongly G -graded ring. A *module frame system* for S is a family $(x_g, y_g)_{g \in G}$ such that for each $g \in G$ there exists a positive integer n_g with

$$x_g \in M_{n_g,1}(S_g), \quad y_g \in M_{n_g,1}(S_{g^{-1}}) \quad \text{and} \quad y_g^t x_g = 1.$$

In addition, $n_e = 1$ and $x_e = y_e = 1$.

Given a module frame system (x_g, y_g) one can write a homogeneous element $s \in S_g$ as

$$s = u^t x_g \quad (\text{choose e.g. } u = s y_g)$$

for $u \in M_{n_g,1}(R)$, regarding the components of the column vector x_g as generators of S_g (as an R -module).

For each $g \in G$, define $\alpha_g : M_{n,m}(S) \rightarrow M_{n_g n, n_g m}(S)$ by

$$\alpha_g(A) = \begin{pmatrix} x_g a_{11} y_g^t & x_g a_{12} y_g^t & \cdots & x_g a_{1m} y_g^t \\ \vdots & \vdots & & \vdots \\ x_g a_{n1} y_g^t & x_g a_{n2} y_g^t & \cdots & x_g a_{nm} y_g^t \end{pmatrix},$$

for $A = (a_{ij}) \in M_{n,m}(S)$. For $s \in S$ one has

$$x_g s = x_g s y_g^t x_g = \alpha_g(s) x_g$$

similar to the case of crossed products. For all $g, h \in G$, define

$$\omega(g, h) := (x_g \triangleright x_h) y_{gh}^t \in M_{n_g n_h, n_{gh}}(S_e),$$

$$\tilde{\omega}(g, h) := x_{gh} (y_h \triangleleft y_g)^t \in M_{n_{gh}, n_g n_h}(S_e),$$

giving

$$x_g \triangleright x_h = \omega(g, h) x_{gh}$$

where $\triangleright, \triangleleft$ denote the left and right Kronecker products of matrices.

Factory system relations

$$x_g s = x_g s y_g^t x_g = \alpha_g(s) x_g$$

$$x_g \triangleright x_h = \omega(g, h) x_{gh}$$

The maps $\alpha, \omega, \tilde{\omega}$ satisfy the following relations

$$\tilde{\omega}(g, h) \omega(g, h) = \alpha_{gh}(1) \quad \text{and} \quad \omega(g, h) \tilde{\omega}(g, h) = \alpha_g(\alpha_h(1)),$$

$$\omega(g, h) \alpha_{gh}(r) = \alpha_g(\alpha_h(r)) \omega(g, h),$$

$$(\omega(g, h) \triangleright 1_{n_k}) \omega(gh, k) = \alpha_g(\omega(h, k)) \omega(g, hk).$$

Compare with the crossed product relations:

$$\alpha_g(\alpha_h(r)) = \omega(g, h) \alpha_{gh}(r) \omega(g, h)^{-1},$$

$$\omega(g, h) \omega(gh, k) = \alpha_g(\omega(h, k)) \omega(g, hk)$$

Definition (Factor system)

Let R be a unital ring and let G be a group. A *factor system* for (R, G) is a triple (n, α, ω) consisting of

- a family $(n_g)_{g \in G}$ of positive integers, normalized by $n_e = 1$;
- a family $(\alpha_g : R \rightarrow M_{n_g}(R))_{g \in G}$ of ring homomorphisms, normalized by $\alpha_e = \text{id}_R$;
- a family $(\omega(g, h) \in M_{n_g n_h, n_{gh}}(R))_{g, h \in G}$ of elements, normalized by $\omega(g, 1) = \omega(1, g) = \alpha_g(1)$ for all $g \in G$;
- a family $(\tilde{\omega}(g, h) \in M_{n_g n_h, n_{gh}}(R))_{g, h \in G}$ of elements;

satisfying

$$\tilde{\omega}(g, h)\omega(g, h) = \alpha_{gh}(1) \quad \text{and} \quad \omega(g, h)\tilde{\omega}(g, h) = \alpha_g(\alpha_h(1)),$$

$$\omega(g, h)\alpha_{gh}(r) = \alpha_g(\alpha_h(r))\omega(g, h),$$

$$(\omega(g, h) \triangleright 1_{n_k})\omega(gh, k) = \alpha_g(\omega(h, k))\omega(g, hk).$$

Factor systems and strongly graded rings

Given a factor system, one can construct a strongly G -graded ring. Conversely, every factor system arises from a strongly G -graded ring.

Theorem

Each factor system for (R, G) gives rise to a strongly G -graded ring with principal component R , for which it appears as an associated factor system.

Defining a notion of factor systems being conjugate, one shows that the conjugacy classes of factor systems precisely correspond to isomorphism classes of strongly graded algebras.

Lifting derivations

Let S be a strongly G -graded ring with principal component $R = S_e$, and let $\delta \in \text{Der}(R)$. We are interested in understanding when δ can be "lifted" to a derivation $\hat{\delta} \in \text{Der}(S)$, i.e. when there exists a derivation $\hat{\delta} : S \rightarrow S$ such that $\delta = \hat{\delta}|_R$.

Note that since S_g is a R -bimodule, every inner derivation of R lifts to an inner derivation of S .

Theorem

Let S be a strongly G -graded ring with principal component R , let (n, α, ω) be a factor system associated with a module frame system $(x_g, y_g)_{g \in G}$ for S , and let $\delta : R \rightarrow R$ be a derivation. Then the following assertions hold:

- Let $\hat{\delta} : S \rightarrow S$ be a graded derivation lifting δ . For each $g \in G$, define

$$\eta(g) := \hat{\delta}(x_g)y_g^t \in M_{n_g, n_g}(R)$$

Then, for all $g \in G$ and $r \in R$,

$$[\delta, \alpha_g](r) = [\eta(g), \alpha_g(r)] + \alpha_g(r)\delta(\alpha_g(1)). \quad (1)$$

and, for all $g, h \in G$,

$$\begin{aligned} \delta(\omega(g, h)) &= (\eta(g) \triangleright 1_{n_h})\omega(g, h) + \alpha_g(\eta(h))\omega(g, h) \\ &\quad + \omega(g, h)(\delta(\alpha_{gh}(1)) - \eta(gh)) \end{aligned} \quad (2)$$

Theorem (cont'd)

- *Conversely, suppose that for each $g \in G$ there exist $\eta(g) \in M_{n_g, n_g}(R)$ with $\eta(e) = 0$ such that (1) holds for all $g \in G$ and $r \in R$, and (2) holds for all $g, h \in G$. Define a map $\hat{\delta} : S \rightarrow S$ by additively extending*

$$\hat{\delta}(u^t x_g) := \delta(u)^t x_g + u^t \eta(g) x_g \quad (3)$$

for all $g \in G$ and $u \in M_{n_g, 1}(R)$. Then $\hat{\delta}$ is a graded lift of δ .

Hence, the obstructions for lifting a derivation is formulated in terms of the factor system.

From the factor system and a derivation of R , one can construct a group 2-cocycle Δ_δ with values in $Z(R)$, whose class turns out to be an obstruction to lifting the derivation.

Theorem

Let S be a strongly G -graded ring with principal component R , let (n, α, ω) be an associated factor system, and let $\delta : R \rightarrow R$ be a derivation. Assume that for each $g \in G$ there exists $\eta(g) \in M_{n_g}(R)$ with $\eta(e) = 0$ such that, for all $r \in R$

$$[\delta, \alpha_g](r) = [\eta(g), \alpha_g(r)] + \alpha_g(r)\delta(\alpha_g(1)).$$

Then δ admits a graded lift to S if and only if $[\Delta_\delta] \in H_\beta^2(G, Z(R))$ vanishes.

Corollary

Let S be a strongly G -graded ring with principal component R , let (n, α, ω) be an associated factor system, and let $\delta : R \rightarrow R$ be a derivation. such that $\delta(\alpha_g(r)) = \alpha_g(\delta(r))$ for all $g \in G$ and $r \in R$. If $[\Delta_\delta] \in H_\beta^2(G, Z(R))$ vanishes, then δ admits a graded lift to S .

Corollary

Let S be a strongly \mathbb{Z} -graded ring with principal component R , let (n, α, ω) be an associated factor system, and let $\delta : R \rightarrow R$ be a derivation. Assume that there exists $\eta \in M_{n_1, n_1}(R)$ such that, for all $r \in R$,

$$[\delta, \alpha_1](r) = [\eta, \alpha_1(r)] + \alpha_1(r)\delta(\alpha_1(1)). \quad (4)$$

Then δ admits a graded lift to S .

For \mathbb{Z} -gradings, the cohomological obstruction vanishes since $H_\beta^n(\mathbb{Z}, M) = 0$ for $n \geq 2$.

Summary

- We have introduced factor systems as a description of strongly graded algebras in terms of the principal component and the grading group.
- Factor systems give an explicit computational framework for strongly graded algebras which also helps in understanding the relation between the principal component and the homogeneous components.
- As an example, we have applied it to the lifting problem of derivations, giving rather concrete conditions for the existence of a lift, related to group cohomology.
- The explicit conditions depend on the choice of factor system, but the end result does not (i.e. can one lift the derivation or not).
- We prove several other results and apply factor systems to Leavitt path algebras, where module frame system are explicitly constructed.

Thank you!