Rigidity and Flexibility of Hypergraphs through Graphs of Groups

Joannes Vermant

Umeå University

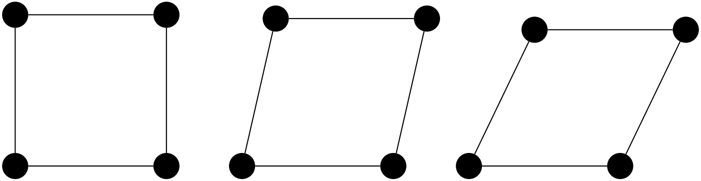
28/05/2026

Rigidity of graphs

A d-dimensional bar-joint framework is a graph $\Gamma = (V, E)$, together with a function $p : V \to \mathbb{R}^d$.

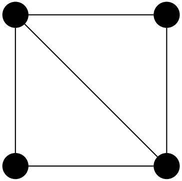
Two bar-joint frameworks (Γ, p) and (Γ, q) are said to be equivalent if for any edge xy, one has

$$d(p(x), p(y)) = d(q(x), q(y)).$$

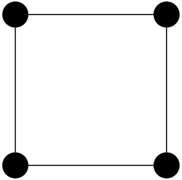


3 equivalent frameworks in the plane.

Rigidity



Rigid



Flexible (= not rigid)

A bar joint framework (Γ, p) is said to be rigid if any bar joint framework (Γ, q) which is equivalent to p, and which is sufficiently close to p, is congruent to p.

For generically chosen coordinates, the rigidity properties only depend on the graph. One can thus talk about rigidity of a graph.

Geiringer-Laman theorem

To characterise rigidity, it suffices to characterise *minimal* rigidity. The generically minimally rigid graphs are those that are generically rigid, but taking away any edge makes the graph generically non-rigid.

Theorem (Geiringer 1927, Laman 1970)

A graph $\Gamma = (V, E)$ *is generically minimally rigid in the plane if and only if*

$$|E| = 2|V| - 3,$$

and for any subgraph $\Gamma' = (V', E')$*, one has*

$$|E'| \leq 2|V'| - 3 \text{ for all } V' \subseteq V.$$

Geiringer-Laman theorem

To characterise rigidity, it suffices to characterise *minimal* rigidity. The generically minimally rigid graphs are those that are generically rigid, but taking away any edge makes the graph generically non-rigid.

Theorem (Geiringer 1927, Laman 1970)

A graph $\Gamma = (V, E)$ *is generically minimally rigid in the plane if and only if*

$$|E| = 2|V| - 3,$$

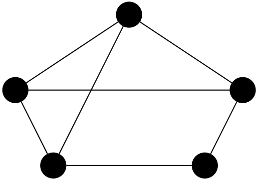
and for any subgraph $\Gamma' = (V', E')$*, one has*

$$|E'| \leq 2|V'| - 3 \text{ for all } V' \subseteq V.$$

In dimensions 3 and up, characterising rigidity is an open problem. There are however, various

- Liftings of scenes and parallel redrawings (Whiteley 1989),
- Panel-hinge frameworks, body-bar frameworks, . . . (Whiteley 1988, Tay 1989).
- Line-constrained frameworks (Cruickshank, Guler, Jackson, Nixon 2018)
- . . .

Example



The conditions on subset are

$$|E[V']| \leq 2|V'| - 3.$$

- The full graph needs to induce 7 edges

$$|E[V]| = 2|V| - 3.$$

- Any subset with 4 vertices induces at most 5 edges.
- Any subset with 3 vertices induces at most 3 edges. (Always OK)
- Any subset with 2 vertices induces at most 1 edge. (Always OK)

Geiringer-Laman theorem: sketch of proof

Theorem (Geiringer 1927, Laman 1970)

A graph $\Gamma = (V, E)$ is generically minimally rigid in the plane if and only if

$$|E| = 2|V| - 3,$$

and for any subgraph $\Gamma' = (V', E')$

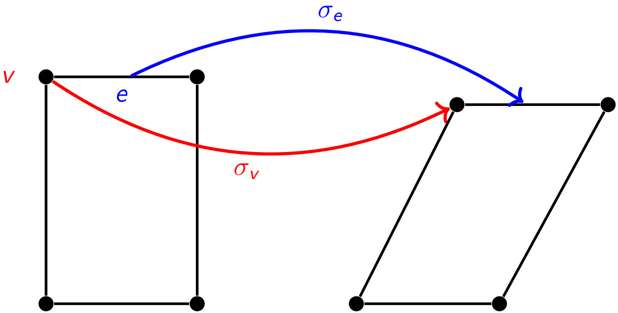
$$|E'| \leq 2|V'| - 3.$$

One proof of the Geiringer Laman theorem goes as follows.

- $\implies$: relatively easy.
- $\impliedby$:
  1. Show that all $(2, 3)$-tight graphs can be built, strating from a single edge, using simple graph operations (extension moves).
  2. Geometric step: One shows that applying an extension move to a minimally rigid graph yields a minimally rigid graph.

Group-theoretic model of rigidity-type problems

Given two equivalent bar joint frameworks $p, q : V \to$ of a graph $\Gamma = (V, E)$, one can find $\sigma_x \in E(2)$ for all $v \in V$, and all $e \in E$ one has



$\sigma_v p(v) = q(v)$,

$\sigma_e p(v) = q(v)$,
$\sigma_e p(w) = q(w)$

where $e = vw$.

The conditions on the previous slide imply

$$\sigma_e^{-1}\sigma_v \in \mathsf{Stab}(p(v)),$$

In fact, given such a collection of group elements, this also determines a equivalent bar-joint framework.

One can then describe bar-joint frameworks by only describing the stabiliser subgroups, and one can then describe all equivalent frameworks using group elements instead.

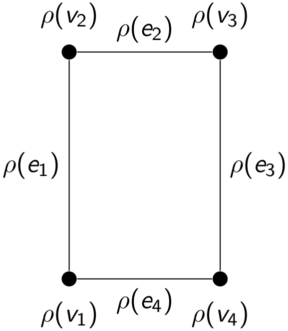
Definition (Stokes, V.)

Let $\Gamma = (V, E)$ be a hypergraph. A **graph-of-groups realisation** in a group G consists of an assignment of a subgroup of G to every vertex $v \mapsto \rho(v) \subseteq G$:

$$v \mapsto \rho(v)$$

Additionally, for each hyperedge, one defines

$$\rho(e) = \bigcap_{v \in e} \rho(v).$$



For the 2-dimensional bar-joint framework from the first slide, each vertex gets $\mathrm{Stab}(p(v))$ associated to it. This is the group of rotations around $p(v)$, which is isomorphic to $O(2)$.

Definition (Stokes, V.)

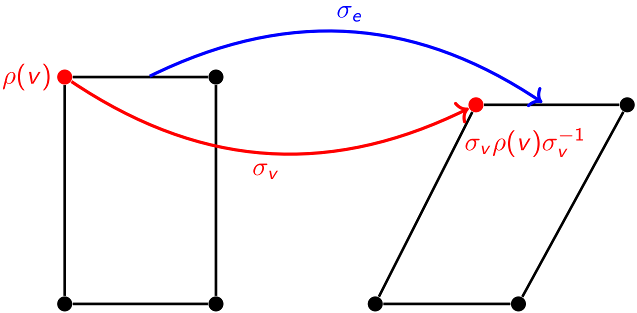
Let ρ be a graph-of-groups realisation in a group G. A motion M of $\rho(\Gamma)$ is a collection of group elements $(\sigma_x)_{x \in V \cup E} \in G^{V \cup E}$ such that

$$\sigma_e^{-1}\sigma_v \in \rho(v) \text{ for all vertex - edge incidences } v \sim e$$

This defines a new graph-of-groups realisation ρ^M, defined by

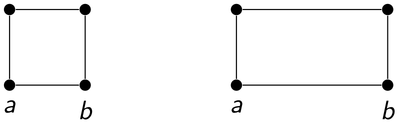
$$\rho^M(v) = \sigma_v \rho(v) \sigma_v^{-1}.$$

This gives graph-of-groups realisations the structure of a groupoid.



Example: parallel redrawings

Given a graph in $\mathbb{R}^2$, determine the ways are there to draw the graph such that all edges retain their direction when considered as line segments in the plane. In other words, the edges should be parallel in equivalent realisations.



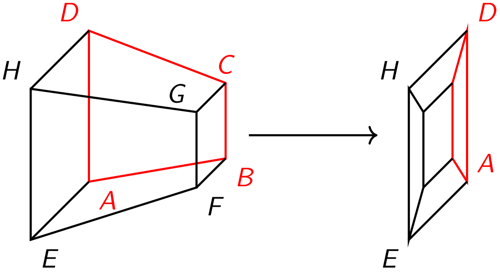
The group G is the group of dilations and translations of $\mathbb{R}^2$. Using homogeneous coordinates, as matrices, the elements of this group are given by:

$$\begin{bmatrix} \lambda & 0 & x \\ 0 & \lambda & y \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } \lambda \in \mathbb{R}^*, x, y \in \mathbb{R}$$

One takes $\rho(v)$ to be the stabilisers of some $p_v \in \mathbb{R}^2$ for each $v \in V$.

Example: scenes

Given a picture in $\mathbb{R}^2$, determine the scenes that project down to a given picture.



G is given by the affine transformations $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\pi \circ f = \pi$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection. As matrices, these are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & z & \lambda \end{bmatrix}, \text{ where } \lambda \in \mathbb{R}^*, x, y, z \in \mathbb{R}$$

$\rho(v)$ is $\mathrm{Stab}(x)$ for some $p_v \in \mathbb{R}^3$.

More examples and general philosophy

Picking the group G determines the "trivial motions" of the structure. The subgroups $\rho(v)$ determine the geometric objects one wants to model.

In each of the examples below, p_v is an element of the space G naturally acts on.

Group	**Subgroups** $\rho(v)$	**Realisations of the (hyper)graph**
$E(d)$	$\mathrm{Stab}(p_v) \cong O(d)$	Euclidean frameworks
$O(d)$	$\mathrm{Stab}(p_v) \cong O(d-1)$	Spherical frameworks
$SL(n, \mathbb{R}) \ltimes \mathbb{R}^n$	$\mathrm{Stab}(p_v) \cong SL(n, \mathbb{R})$	Volume frameworks
S_n	S_{n-1}	Proper colourings of graphs
...	...	...

Infinitesimal motions

When G is a Lie group and $\rho(v)$ is a closed Lie subgroup for every vertex $v \in V$, one can define local and infinitesimal rigidity.

Linearising the condition for a motion at the identity for the Lie group gives

$$\begin{array}{c} \sigma_e^{-1}\sigma_v \in \rho(v) \text{ for all incidences } v \sim e \\ \downarrow \\ -W_e + W_v \in \mathfrak{h}_v \text{ for all incidences } v \sim e, \end{array}$$

where $\mathfrak{h}_v$ is the Lie algebra of $\rho(v)$.

Infinitesimal motions

Since a motion $M = (\sigma_x)_{x \in V \cup E}$ does not change ρ (meaning $\rho^M = \rho$) if $\sigma_x \in \rho(x)$ for all $x \in V \cup E$, we additionally want to consider $W_v = W_v'$ if $W_v = W_v' \mod \mathfrak{h}_v$, i.e. we define the vector space of infinitesimal motions IM_ρ to be the space

$$(\overline{W}_x)_{x \in V \cup E} \in \prod_{x \in V \cup E} \mathfrak{g}/\mathfrak{h}_x$$

such that

$$W_e - W_v \in \mathfrak{h}_v \text{ for all vertex-edge incidences } v \sim e.$$

One alway has the trivial infinitesimal motions in the image:

$$i : \mathfrak{g} \to IM_\rho : W \mapsto \left(\overline{W}, \ldots, \overline{W}\right)$$

Classes of well-behaved graph-of-groups realisations

We consider the graph-of-groups realisations of a hypergraph Γ such that each $\rho(v)$ is conjugate to some fixed closed Lie subgroup H of a group G, i.e. for all $v \in V$ there exists a $g_v \in G$ such that

$$\rho(v) = g_v H g_v^{-1}.$$

One can identify $(G/N_G(H))^V$ with the stated class of graph-of-groups realisations.

For example, one has:

$$G = E(d), H = O(d) \rightarrow \ \rho \text{ corresponding to bar-joint frameworks in } \mathbb{R}^d.$$
$$G = O(d+1), H = O(d) \rightarrow \ \rho \text{ associated to graph frameworks on } S^d.$$
$$G = PSL(2, \mathbb{R}), H = SO(2) \rightarrow \ \rho \text{ corresponding to graph frameworks on } \mathbb{H}$$

Main theorem

For any graph Γ and $a \in \mathbb{N} \setminus \{0\}$, we denote by $a \cdot \Gamma$ the multi-graph with a copies of each edge.

We recall that a graph $\Gamma = (V, E)$ is (d, ℓ)-sparse if for every subset $V' \subseteq V$, one has

$$|E[V']| \leq d|V| - \ell.$$

We say a graph-of-groups realisation is independent, if every edge removes removes the combinatorially expected degrees of freedom.[1]

Theorem (V. 26+)

Let Γ be a graph. Let G be a Lie group, and suppose that $H \subseteq G$ is a 1-dimensional connected subgroup such that $N_G(H)/H$ is finite. Then, for any graph Γ, there is a dense open subset of $\rho \in (G/N_G(H))^{|V|}$ such that $\rho(\Gamma)$ is independent if and only if $(\dim(G) - 2) \cdot \Gamma$ is $(\dim(G) - 1, \dim(G)) -$sparse.

The proof is an inductive proof, based on a result by Frank and Szegő (2003).

---

[1] This is not the actual definition.

Main theorem

Theorem (V. 26+)

Let Γ be a graph. Let G be a Lie group, and suppose that $H \subseteq G$ is a 1-dimensional connected subgroup such that $N_G(H)/H$ is finite. Then, for any graph Γ, there is a dense open subset of $\rho \in (G/N_G(H))^{|V|}$ such that $\rho(\Gamma)$ is independent if and only if $(\dim(G) - 2) \cdot \Gamma$ is $(\dim(G) - 1, \dim(G))-$sparse.

- $G = E(2), H = O(2)$ gives the Geiringer-Laman theorem.
- $G = PSL(2, \mathbb{R}), H = SO(2)$ gives a characterisation of infinitesimal rigidity of graphs on the hyperbolic plane $\mathbb{H}$ and $G = O(3), H = O(2)$ gives the characterisation on S^2.
- Taking G to be the $(d + 1)-$dimensional group of dilations in $\mathbb{R}^d$, and H to be a 1 dimensional subgroup of dilations gives a characterisation of parallel rigidity of graphs in $\mathbb{R}^d$ (Whiteley 1996)

Thank you for listening!

- Klara Stokes, Joannes Vermant, *Structural rigidity and flexibility using graphs of groups* Applicable Algebra in Engineering, Communication and Computing (2026)
- Joannes Vermant, *Homological methods in rigidity theory using graphs of groups* `arXiv:2603.05435v2`