Central idempotents in group-graded rings

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Reference

This talk is based on my recent preprint:

- *Central idempotents in group-graded rings*,
  arXiv:2605.20008 [math.RA] (2026)

Convention

- Today, “a ring” means “an associative ring that is not necessarily unital”.

Outline

Group rings

Definition

Given a unital ring A and a group G, the *group ring* $A[G]$ is defined as the free left (and right) A-module with $\{u_g\}_{g \in G}$ as its basis. Multiplication is defined by extending the rule

$$(au_g)(bu_h) := ab\ u_{gh},$$

for $a, b \in A$ and $g, h \in G$.

- Each element of $A[G]$ can be written on the form $r = \sum_{g \in G} a_g u_g$.
- We write $\mathrm{Supp}(r) := \{g \in G \mid a_g \neq 0\}$.
- The *support group* of a nonzero element r is the subgroup of G generated by the set $\mathrm{Supp}(r)$.

Example: An idempotent with finite support group

Consider the group ring $R := M_2(\mathbb{C})[\mathbb{Z}]$.

Example

Set

$$f := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_0.$$

Note that:

- $f = f^2$, that is f is idempotent.
- $\mathrm{Supp}(f) = \{0\}$.
- The support group of f is finite!

Example: An idempotent with infinite support group

Consider the group ring $R := M_2(\mathbb{C})[\mathbb{Z}]$.

Example

Set

$$f := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_0 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} u_1.$$

One can check that:

- $f = f^2$, that is f is idempotent.
- $\mathrm{Supp}(f) = \{0, 1\}$.
- The support group of f is infinite!
- $f \notin Z(R)$.

1964: Commutative group rings

Theorem (W. Rudin & H. Schneider)

Let A be a unital commutative ring and let G be an abelian group. If f is a nonzero idempotent in the group ring $A[G]$, then f has finite support group.

Figure: W. Rudin

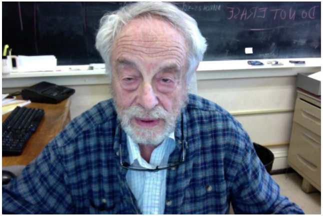

Figure: H. Schneider

Reference: W. Rudin and H. Schneider, Idempotents in group rings, *Duke Math. J.* 31 (1964), 585–602.

1970: Non-commutative group rings

Theorem (R. G. Burns)

Let A be a unital ring and let G be a group. If f is a nonzero central idempotent in the group ring $A[G]$, then f has finite support group.

Figure: A different Burns

Reference: R. G. Burns, Central idempotents in group rings, *Canad. Math. Bull.* 13 (1970), 527–528.

A historical remark

Remark

Burns's result was actually proven for non-unital group rings!

Remark

Independently, A. A. Bovdi and S. V. Mihovski proved a partial generalization of Burns's result for (unital) G-crossed products.

References:

- A. A. Bovdi and S. V. Mihovski, Idempotents of crossed products, *Dokl. Akad. Nauk SSSR* 195 (1970), 263–265.
- A. A. Bovdi and S. V. Mihovski, Algebraic elements of crossed products, Rings, modules and radicals (Proc. Colloq., Keszthely, 1971), pp. 103–116, *Colloq. Math. Soc. János Bolyai*, Vol. 6 North-Holland Publishing Co., Amsterdam-London (1973).

1976: Commutative group rings, again

Theorem (H. Bass)

Let A be a unital commutative ring and let G be an abelian group. If f is an idempotent in the group ring $A[G]$, then $f \in A[T]$, where T is the torsion subgroup of G.

Figure: H. Bass

Reference: H. Bass, Euler characteristics and characters of discrete groups, *Invent. Math.* 35 (1976), 155–196.

2014: Bass in popular culture ...

2026: Group-graded rings!

Let G be a group with identity element e.

Definition

A (possibly non-unital) ring R is said to be *G-graded* if there is a collection of additive subgroups $\{R_g\}_{g \in G}$ of R such that

- $R = \oplus_{g \in G} R_g$,
- $R_g R_h \subseteq R_{gh}$, for all $g, h \in G$.

If $R_g R_h = R_{gh}$ holds for all $g, h \in G$, then R is called *strongly G-graded*.

Remark

- R_e is a subring of R.
- Every element $r \in R$ can be written on the form $r = \sum_{g \in G} r_g$ with $r_g \in R_g$ zero for all but finitely many $g \in G$.
- We define $\mathrm{Supp}(r) := \{g \in G \mid r_g \neq 0\}$.

Our first main result

Theorem

Let G be an abelian group and let R be a G-graded ring.

- *If f is a nonzero central idempotent in R, then f has finite support group.*
- *In particular, the support of a nonzero central idempotent in R is contained in the torsion subgroup of G.*

Remark

This generalizes Bass's result.

A word about unitality

Remark

Suppose that R is a G-graded ring. Any central idempotent in R can be viewed as a central idempotent in a unital G-graded ring!

- Let D be the Dorroh unitization of R. Recall that $D := R \times \mathbb{Z}$ is an associative and unital ring with addition defined componentwise and multiplication defined by $(r, n)(s, m) := (rs + ns + mr, nm)$.
- $\psi : R \to D, r \mapsto (r, 0)$ is an injective ring homomorphism.
- If f is a nonzero central idempotent in R, then $\psi(f)$ is a nonzero central idempotent in D.
- D inherits a natural G-grading from R. Indeed, define $D_e := R_e \times \mathbb{Z}$ and $D_g := R_g \times \{0\}$ for every $g \in G \setminus \{e\}$. Then D becomes a unital G-graded ring.
- The support group of f in the G-graded ring R equals the support group of $(f, 0)$ in the unital G-graded ring D.

A little trick

Lemma

Let G be a group and let R be a unital G-graded ring. Equip the group ring $R[G]$ with its canonical G-grading. There is an injective identity-preserving ring homomorphism $\phi : R \to R[G]$ satisfying $\mathrm{Supp}(r) = \mathrm{Supp}(\phi(r))$ for every $r \in R$.

Proof.

For any $g \in G$ and $r_g \in R_g$ we define

$$\phi(r_g) := r_g u_g$$

and extend it to all of R. This map has the desired properties. □

The proof of Theorem A

Proof of Theorem A.

Let f be a nonzero central idempotent in R.

- Embed R into its Dorroh unitization if necessary. Recall that the support of f is preserved!
- Embed f into the group ring $R[G]$ and consider its image $\phi(f)$, which is a nonzero idempotent. Recall that the support of f is preserved!
- Using that G is abelian, we can show that $\phi(f) \in Z(R[G])$.
- By Burns's theorem, $\phi(f)$ has finite support group.

Hence, f has finite support group.

Since G is abelian, the set of torsion elements forms a subgroup T of G, and clearly the support group of f is contained in T. □

Our second main result

Theorem

Let G be a group and let R be a G-graded ring. Suppose that at least one of the following two assertions holds:

1. $R_g r \neq \{0\}$ *for all nonzero homogeneous* $r \in R$ *and all* $g \in \mathrm{Supp}(R)$.
2. $r R_g \neq \{0\}$ *for all nonzero homogeneous* $r \in R$ *and all* $g \in \mathrm{Supp}(R)$.

If f is a nonzero central idempotent in R, then f has finite support group.

Remark

This generalizes Burns's result.

Remark

We can apply this to e.g. s-unital strongly G-graded rings, crystalline G-graded rings, and non-degenerately G-graded rings with R_e prime.

A key lemma

A group is said to be an *FC group* if each of its conjugacy classes is finite.

Lemma

Let G be a group and let R be a G-graded ring. Suppose that at least one of the following two assertions holds:

1. $R_g r \neq \{0\}$ *for all nonzero homogeneous* $r \in R$ *and all* $g \in \mathrm{Supp}(R)$.
2. $r R_g \neq \{0\}$ *for all nonzero homogeneous* $r \in R$ *and all* $g \in \mathrm{Supp}(R)$.

If f is a nonzero central element in R, then the subgroup of G generated by $\mathrm{Supp}(f)$ is an FC group.

FC groups

Recall that a group is said to be *locally normal* if every finite subset is contained in a finite normal subgroup.

Lemma (Burns)

Any FC group G is isomorphic to a subdirect product of a torsion-free abelian group A with a locally normal group B.

Remark

After identifying G with its image in $A \times B$, there are surjective group homomorphisms $\pi_A : G \to A,\ (a,b) \mapsto a$ and $\pi_B : G \to B,\ (a,b) \mapsto b$. Consider $N_A := \ker(\pi_A)$ which is a normal subgroup of G. By the first isomorphism theorem, we have $G/N_A \cong A$. Also note that N_A is isomorphic to a subgroup of B.

Switching gradings

Let R be a G-graded ring. For any non-empty subset X of G, we will write

$$R_X := \oplus_{x \in X} R_x.$$

- If H is a subgroup of G, then R_H is an H-graded subring of R.
- If N is a normal subgroup of G, then we may view R as a G/N-graded ring. For any $C \in G/N$ we set $R_C := \oplus_{x \in C} R_x$. It is easy to see that

  $$R = \oplus_{C \in G/N} R_C, \quad \text{and} \quad R_C R_D \subseteq R_{CD},$$

  for all $C, D \in G/N$. Note, in particular, that N is the identity element of G/N.
- When we view R as a G-graded ring, the so-called *principal component* will be the subring R_e. But when viewed as a G/N-graded ring, the principal component will be the subring R_N.

The proof of Theorem B

Proof.

Let f be a nonzero central idempotent in R, and let H be the support group of f.

- f is contained in the H-graded subring R_H.
- By the key lemma, H is an FC group.
- By Burns's lemma, H is a subdirect product of A and B, where A is torsion-free abelian and B is locally normal.
- There is a normal subgroup N_A of H such that $H/N_A \cong A$.
- We can view R_H as an A-graded ring with principal component R_{N_A}.
- Using that f is central in R_H, Theorem A yields $f \in R_{N_A}$, because A is torsion-free abelian.
- Note that N_A is locally normal. Hence the support of f is contained in a finite subgroup and so is its support group.

□

An interesting example

Example

Let $G := D_\infty = \langle s, t \mid s^2 = t^2 = e \rangle$ be the infinite dihedral group, and let $R := \mathbb{Q}^4$ be the commutative direct product $\mathbb{Q}$-algebra. Set

$$a := (1, 1, -1, -1), \quad b := (1, -1, 0, 0), \quad \text{and} \quad c := (0, 0, 1, -1).$$

Note that $\{1_R, a, b, c\}$ is a basis for R as a $\mathbb{Q}$-vector space.
We define additive subgroups of R as follows:

- $R_e := \mathbb{Q}1_R \oplus \mathbb{Q}a$,
- $R_s := \mathbb{Q}b$,
- $R_t := \mathbb{Q}c$, and
- $R_g := \{(0, 0, 0, 0)\}$ for $g \in G \setminus \{e, s, t\}$.

It is easy to verify that R is G-graded!

An interesting example, continued

Example

Consider the element

$$f := \frac{1}{2}1_R + \frac{1}{2}b + \frac{1}{2}c = \frac{1}{2}(1,1,1,1) + \frac{1}{2}(1,-1,0,0) + \frac{1}{2}(0,0,1,-1) = (1,0,1,0)$$

Note that:

- f is a nonzero central idempotent in R.
- The subgroup generated by $\mathrm{Supp}(f) = \{e, s, t\}$ is equal to the infinite group D_∞.

Remark

- D_∞ is non-abelian and thus violates the requirement in Theorem A.
- $bc = 0$ which leads to $R_s c = \{0\}$ and $cR_s = \{0\}$, preventing the requirement in Theorem B to be fulfilled.

Semigroup-graded rings

A semigroup S is said to be *right reversible* if $Sa \cap Sb \neq \emptyset$ for all $a, b \in S$.

Proposition

Let S be a cancellative semigroup and let R be an S-graded ring. Furthermore, suppose that at least one of the following assertions holds:

- a. *S is commutative.*
- b. *S is right (or left) reversible, and at least one of the following two assertions holds:*
    - i. *$R_g r \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathrm{Supp}(R)$.*
    - ii. *$r R_g \neq \{0\}$ for all nonzero homogeneous $r \in R$ and all $g \in \mathrm{Supp}(R)$.*

If f is a nonzero central idempotent in R, then f has finite support semigroup in S.

Leavitt path rings

Proposition

Let A be a unital ring, let E be a directed graph, and let $R := L_A(E)$ be the corresponding Leavitt path ring. Equip R with its canonical $\mathbb{Z}$-grading. If f is a central idempotent in R, then $f \in R_0$.

Partial skew group rings

Proposition

Let G be a torsion-free abelian group and let $(\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$ be a partial action of G on an s-unital ring A, with D_g s-unital for every $g \in G$. Every central idempotent in the partial skew group ring $A \star_\alpha G$ is contained in A.

Proposition

Let G be a group and let $(\{\alpha_g\}_{g \in G}, \{D_g\}_{g \in G})$ be a partial action of G on an s-unital prime ring A, with D_g s-unital for every $g \in G$. Every nonzero central idempotent in the partial skew group ring $A \star_\alpha G$ has finite support group.

Bonus material: ICC groups

In an *ICC group*, each non-trivial element has an infinite conjugacy class.

Remark

Non-trivial abelian groups cannot be ICC. But there exist ICC groups with torsion!

We get the following corollary from the proof of the key lemma:

Corollary

Let G be an ICC group and let R be a G-graded ring with $\mathrm{Supp}(R) = G$. Suppose that at least one of the following two assertions holds:

1. $R_g r \neq \{0\}$ *for all nonzero homogeneous* $r \in R$ *and all* $g \in \mathrm{Supp}(R)$.
2. $r R_g \neq \{0\}$ *for all nonzero homogeneous* $r \in R$ *and all* $g \in \mathrm{Supp}(R)$.

If f is a central idempotent in R, then $f \in R_e$.

The end

THANK YOU FOR YOUR ATTENTION!