Constructing Koszul filtrations

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Joint work with
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What is a Koszul algebra?

$A = \bigoplus_{i \geq 0} A_i$ a standard graded algebra over a field $\mathbb{K}$.

$A = \mathbb{K}\langle x_1, \ldots, x_n \rangle / J$ two sided homogeneous ideal J

Commutative setting:

$A = \mathbb{K}[x_1, \ldots, x_n] / J$ homogeneous ideal J

$x_1, \ldots, x_n$ have degree 1.

Minimal free resolution of $\mathbb{K}$ as an A-module

$$\cdots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} A \longrightarrow \mathbb{K} \longrightarrow 0$$

A is <u>Koszul</u> if the resolution is linear. [Priddy 1970]

If the F_i's finitely generated: φ_i is a matrix with linear entries.

<u>Baby example</u>. $A = \mathbb{K}[x]/(x^d)$

$$\cdots \xrightarrow{\cdot x^{d-1}} A \xrightarrow{\cdot x} A \xrightarrow{\cdot x^{d-1}} A \xrightarrow{\cdot x} A \to \mathbb{K} \to 0$$

Koszul iff $d=2$.

In general: If A is Koszul then J is generated in degree 2.

Some examples of Koszul algebras

- Polynomial ring $\mathbb{K}[x_1, \dots, x_n]$
- Free associative algebra $\mathbb{K}\langle x_1, \dots, x_n \rangle$
- Exterior algebra on $x_1, \dots, x_n$
- Complete intersections $\mathbb{K}[x_1, \dots, x_n]/(f_1, \dots, f_r)$
- Coordinate ring of s points in $\mathbb{P}^n$ in general position, if $s \leq 2n$ [Kempf 1992]
- Toric rings that are normal and of minimal multiplicity [Herzog-Reiner-Welker 1998]
- Chow rings of matroids. [Mastroeni-McCullough 2022]

How to prove that an algebra is Koszul?

<u>Example</u> [Roos 1993] $A = K[x_1, \ldots, x_6]/J$

$$J = (x_1^2,\ x_1x_2,\ x_2x_3,\ x_3^2,\ x_3x_4,\ x_4^2,\ x_4x_5,\ x_5x_6,\ x_6^2,$$
$$x_1x_3 + \lambda x_3x_6 - x_4x_6,\ x_1x_4 + x_3x_6 + (\lambda-2)x_4x_6) \qquad \lambda \in \mathbb{N}$$

Not Koszul.

non-linear

$$\cdots \longrightarrow F_{\lambda+1} \xrightarrow{\varphi} \underbrace{F_\lambda \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow A}_{\text{linear}} \longrightarrow K \longrightarrow 0$$

How to prove that an algebra is Koszul?

1. Theorem [Fröberg 1975, Anick 1986]

Let R be a polynomial ring, free associative algebra, or exterior algebra.
If J has a quadratic Gröbner basis, then R/J is Koszul.

$A = R/J$ is called G-quadratic if J has a quadratic Gröbner basis after a change of coordinates.

2. Koszul filtration $\quad A = K[x_1, \ldots, x_n]/J$

Family $\mathcal{F}$ of ideals of A such that

- each ideal in $\mathcal{F}$ is generated by linear forms
- $(x_1, \ldots, x_n) \in \mathcal{F}$
- $0 \neq I \in \mathcal{F} \implies \exists\ J \in \mathcal{F}$ such that
  - $I = J + (\ell) \quad \ell \notin J$
  - $J : I \in \mathcal{F}$

Observation.
- $\forall\ 0 \leq t \leq n$ $\mathcal{F}$ contains an ideal with t generators
- $0 \in \mathcal{F}$

<u>Theorem</u> [Conca-Trung-Valla 2001]. Suppose $\mathcal{F}$ is a Koszul filtration of A. Then for each $I \in \mathcal{F}$ A/I is Koszul.
In particular A is Koszul.

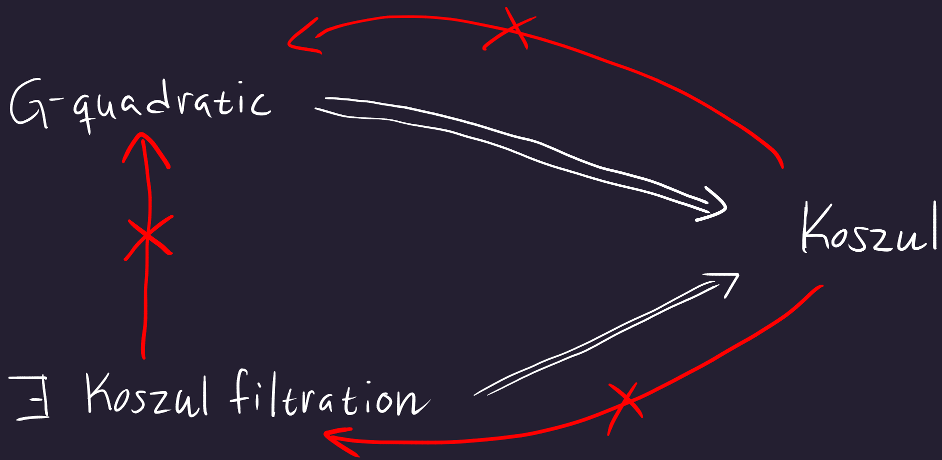
$$
\begin{tikzcd}
\text{G-quadratic} \arrow[dr, Rightarrow] & \\
& \text{Koszul} \arrow[ul, bend right, "\times"'] \arrow[dl, bend left, "\times"] \\
\exists\ \text{Koszul filtration} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

A Koszul algebra which is not G-quadratic and does not have a Koszul filtration.

$\mathbb{C}[x_1, \dots, x_5]/(f_1, \dots, f_5)$ complete intersection

5 generic quadratic forms

$(0):(\ell)$ is always generated in degree > 1.


G-quadratic
Koszul
∃ Koszul filtration

An algebra with a Koszul filtration which is not G-quadratic

$$\mathbb{C}[x_1, x_2, x_3, x_4] / (x_1x_3,\ x_1x_4,\ x_1x_2 - x_2x_4,\ x_1^2 + x_2x_3,\ x_2^2)$$

No quadratic monomial ideal with the same Hilbert series.

Koszul filtration:

$$(x_1, x_3, x_4) : (x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)$$
$$(x_3, x_4) : (x_1, x_3, x_4) = (x_1, x_2, x_3, x_4)$$
$$(x_3) : (x_3, x_4) = (x_1, x_3)$$
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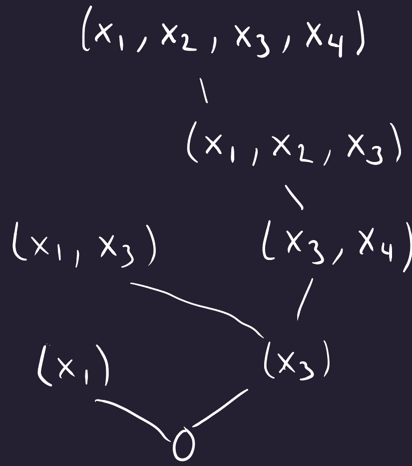
$$(0) : (x_1) = (x_3, x_4)$$
$$(0) : (x_3) = (x_1)$$



$$
\begin{tikzcd}
\text{G-quadratic} \arrow[dr, Rightarrow] \arrow[dd, shift right=2, squiggly, "?"'] & \\
& \text{Koszul} \arrow[ul, bend right, "\times" marking] \arrow[dl, bend left, "\times" marking] \\
\exists\ \text{Koszul filtration} \arrow[ur, Rightarrow] \arrow[uu, shift right=2, "\times" marking] &
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\end{tikzcd}
$$

A G-quadratic algebra with no Koszul filtration.

$$\mathbb{K}[a,b,c,d,e,f,g,h,i,j,k,l]/I$$

$$I = (a^2, \ldots, l^2,\ ab-aj,\ ab-bj,\ ac-ad,\ ac-cd,\ cg-cj,\ cg-gj,\ bd-bg,\ bd-dg,$$
$$de-df,\ de-ef,\ gh-gi,\ gh-hi,\ jk-jl,\ jk-kl)$$

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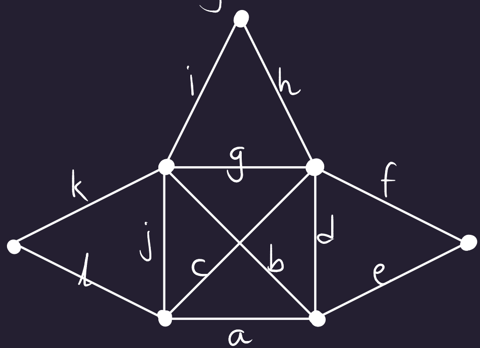
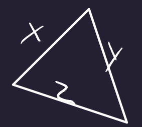
Möbius algebra





$xy - xz$

$xy - yz$

$xz - yz$

Quadratic Gröbner basis for the Möbius algebra, under certain conditions on the graph.

[LaClair, Mastroeni, McCullough, Peeva. 2025]

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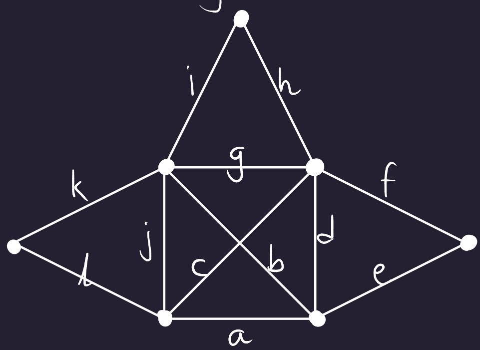

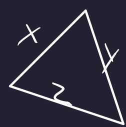

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Lex. $a > b > c > e > h > k > l > i > f > j > g > d$ (independent of $\mathbb{K}$)

$$\mathbb{K}[a,b,c,d,e,f,g,h,i,j,k,l]/I$$

$$I = (a^2, \ldots, l^2,\ ab-aj,\ ab-bj,\ ac-ad,\ ac-cd,\ cg-cj,\ cg-gj,\ bd-bg,\ bd-dg,\ de-df,\ de-ef,\ gh-gi,\ gh-hi,\ jk-jl,\ jk-kl)$$

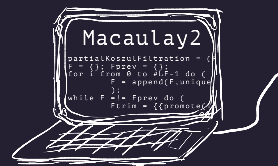
If $\mathbb{K} = \mathbb{F}_2$ then there is a Koszul filtration:

(0), (d), $(d+e+f)$, (d,e), $(b+e+f+g, d+e+f)$, $(d+e+f, g)$, (d,e,f),
$(a+c+e+f, b+e+f+g, d+e+f)$, $(b+e+f, d+e+f, g)$, $(b+c+f, d+e+f, g)$,
$(a+c, b+g, d, e+f)$, $(b+e+f, c+j, d+e+f, g)$, $(b, d, e+f, g)$, $(a+c, b+g, d, e+f, j)$,
$(a+g+j, b+g, c+g+j, d, e+f)$, $(b+e+f, c+j, d+e+f, g, h+i)$, $(a, b+g, c, d, e+f, j)$,
$(a+g, b+g, c+g, d, e+f, j)$, $(a+j, b, c+j, d, e+f, g)$, $(b+e+f, c+j, d+e+f, g, h, i)$,
$(b, c+j, d, e+f, g, h+i)$, $(a, b, c, d, e+f, g, j)$, $(a+g, b+g, c+g, d, e+f, j, k+l)$,
$(a+j, b, c+j, d, e+f, g, h+i)$, $(a, b, c, d, e+f, g, h+i, j)$, $(a+g, b+g, c+g, d, e+f, j, k, l)$,
$(a, b, c, d, e+f, g, h, i, j)$, $(a, \ldots, j)$, $(a, \ldots, k)$, $(a, \ldots, l)$.

$$\mathbb{K}[a,b,c,d,e,f,g,h,i,j,k,l]/I$$

$$I = (a^2, \ldots, l^2,\ ab-aj,\ ab-bj,\ ac-ad,\ ac-cd,\ cg-cj,\ cg-gj,\ bd-bg,\ bd-dg,\ de-df,\ de-ef,\ gh-gi,\ gh-hi,\ jk-jl,\ jk-kl)$$

If $\mathbb{K} = \mathbb{F}_3$ there is no Koszul filtration.



Proof idea

1. If $(0):(L)$ is nonlinear then (L) cannot belong to a Koszul filtration
2. If $(0):(L)$ has a nonlinear resolution in the first 5 steps then (L) cannot belong to a Koszul filtration
3. If $I=(L_1,\ldots,L_s)$ has a nonlinear resolution in the first 5 steps then I cannot belong to a Koszul filtration

265.720 linear forms, removed scalar multiples.

Apply 1 and 2 $\rightsquigarrow$ 30 linear forms L s.t. (L) can potentially be part of a Koszul filtration.

$$d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ d+e,\ d-e,\ d+f,\ d-f,\ e+f,\ g+h,\ g-h,\ g+i,\ g-i,\ h+i,$$
$$j+k,\ j-k,\ j+l,\ j-l,\ k+l,\ d+e+f,\ d-e-f,\ g+h+i,\ g-h-i,\ j+k+l,\ j-k-l$$

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$j+k,\ j-k,\ j+l,\ j-l,\ k+l,\ d+e+f,\ d-e-f,\ g+h+i,\ g-h-i,\ j+k+l,\ j-k-l$

Suppose (d) would be part of a Koszul filtration.

$$(0):(d) = (a-c,\ b-g,\ d,\ e-f)$$

Must include a chain of ideals

$$(d) = I_1 \subset I_2 \subset I_3 \subset I_4 = (a-c,\ b-g,\ d,\ e-f)$$

Go through all possible I_2, I_3 and compute $I_t : I_{t+1}$

Have to include one of

(a,c,d), $(a+g+j,\ b+c+j,\ d)$, $(a,c,d,e-f)$, $(a+g+j,\ b+c+j,\ d,\ e-f)$.

All have nonlinear resolutions (computing 5 steps).

$\rightsquigarrow$ (d) cannot be part of a Koszul filtration.

~~30~~ 29 linear forms L s.t. (L) can potentially be part of a Koszul filtration.

$\cancel{d}$, $e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ d+e,\ d-e,\ d+f,\ d-f,\ e+f,\ g+h,\ g-h,\ g+i,\ g-i,\ h+i,$
$j+k,\ j-k,\ j+l,\ j-l,\ k+l,\ d+e+f,\ d-e-f,\ g+h+i,\ g-h-i,\ j+k+l,\ j-k-l$

Similar arguments apply to the remaining linear forms.

Conclusion :

$$\mathbb{F}_3[a,b,c,d,e,f,g,h,i,j,k,l]/I$$

$$I = (a^2, \ldots, l^2,\ ab-aj,\ ab-bj,\ ac-ad,\ ac-cd,\ cg-cj,\ cg-gj,\ bd-bg,\ bd-dg,$$
$$de-df,\ de-ef,\ gh-gi,\ gh-hi,\ jk-jl,\ jk-kl)$$

is G-quadratic, but does not have a Koszul filtration.

Thank you!