Separable functors and firm modules

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Talk based on:

- P. Lundström, Separable functors and firm modules, arXiv:2602.13417 (2026)

Outline

- Maschke's theorem
- Modules
- Separable ring extensions
- Separable functors
- Simple and semisimple objects in categories

Theorem (Maschke 1899)

Let G be a finite group. Suppose that K is a field whose characteristic does not divide the order $|G|$ of G. Then the group ring $K[G]$ is semisimple.

Proof

Consider $A := K[G]$ as a left module over itself. Let V be an A-submodule of A. Let π be any K-vector space projection of A onto V. Consider the map $\overline{\pi} : A \to V$ where

$$\overline{\pi}(a) := \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot a)$$

for $a \in A$. Then $\overline{\pi}$ is an A-module projection onto V. Therefore $A = V \oplus \ker(\overline{\pi})$. □

Theorem (Connell 1963)

Let G be a finite group and B an associative unital ring. Suppose that B is semisimple and $|G|$ is invertible in B. Then the group ring $B[G]$ is semisimple.

Generalizations

- Crossed products defined by groups (Montgomery, Lorenz, Passman 1980)
- Strongly group-graded rings (Năstăsescu 1983)
- Strongly groupoid-graded rings (· 2007)
- Epsilon-strongly group-graded rings (·, Öinert and Pinedo 2018)

Question

Is there a version of Maschke's theorem that holds for group rings over nonunital rings?

Theorem A

Let G be a finite group and B an associative ring with local units. Suppose that B is semisimple and $|G|B = B$. Then the group ring $B[G]$ is semisimple.

Proof based on separable functors (Năstăsescu, Van den Bergh, Van Oystaeyen 1989)

- Suppose that $f : B \to A$ is a homomorphism of unital rings with B be semisimple.
- We want to show that A is semisimple.
- Enough to prove that every left A-module ${}_AM$ is semisimple.
- Consider the restriction functor $\mathrm{Res}_f : {}_A\mathrm{Mod} \ni {}_AM \mapsto {}_BM \in {}_B\mathrm{Mod}$
- Since B is semisimple, $\mathrm{Res}_f({}_AM) = {}_BM$ is semisimple.
- We are done if we can show that Res_f reflects semisimple objects.
- Separable functors reflect semisimple objects! (for abelian categories)
- We are done if we can show that Res_f is a separable functor.

Definition

- Let A and B be rings (associative but not necessarily unital).
- Let M be a left A-module (additive group with a biadditive map $A \times M \ni (a, m) \mapsto am \in M$ with $(aa')m = a(a'm)$ for $a, a' \in A$ and $m \in M$).
- Categories: ${}_A\mathrm{Mod}$, Mod_B, ${}_A\mathrm{Mod}_B$, ${}_A\mathrm{Mod}_A$, ${}_B\mathrm{Mod}_B$
- M is called *unital* if there is $a \in A$ such that $am = m$ for every $m \in M$.
- AM = the set of all finite sums of elements of the form am for $a \in A$ and $m \in M$.
- M is called unitary if $AM = M$.
- M is called firm if the canonical map $A \otimes_A M \to M$ is an isomorphism in ${}_A\mathrm{Mod}$.
- M is called s-unital if $m \in Am$ for every $m \in M$.
- A is called left unital (s-unital, unitary, firm) if it is unital (s-unital, unitary, firm) as a left A-module.
- A is called a ring with local units if for each finite subset $X \subseteq A$ there exists an idempotent $e \in A$ such that $ex = x = xe$ for all $x \in X$.

Example

Let K be a field. Consider the nonunital ring $K^{(\mathbb{N})}$ consisting of all sequences $(k_n)_{n\in\mathbb{N}}$, where $k_n \in K$, and all but finitely many $k_n = 0$, with pointwise addition and multiplication. Then $K^{(\mathbb{N})}$ has local units.

Example

Let K be a field. Consider the nonunital ring $\mathrm{FM}_{\mathbb{N}}(K)$ of matrices with only finitely many nonzero entries in total. This ring is simple and has local units, since any finite set of matrices is supported on finitely many rows and columns, and the diagonal idempotent corresponding to those rows and columns acts as a local unit for that set.

Remark

M unital $\Longrightarrow$ M s-unital $\Longrightarrow$ M unitary and M unital $\Longrightarrow$ M firm $\Longrightarrow$ M unitary

Definition

- ${}_A\mathrm{Mod}^1$: the category of unital left A-modules
- ${}_A\mathrm{UMod}$: the category of unitary left A-modules
- ${}_A\mathrm{FMod}$: the category of firm left A-modules
- ${}_A\mathrm{SMod}$: the category of s-unital left A-modules

Proposition

- *A left unital* $\Longrightarrow$ ${}_A\mathrm{Mod}^1 = {}_A\mathrm{UMod} = {}_A\mathrm{FMod} = {}_A\mathrm{SMod}$
- *A left s-unital* $\Longrightarrow$ ${}_A\mathrm{UMod} = {}_A\mathrm{FMod} = {}_A\mathrm{SMod}$

Definition

- A a left s-unital ring and $M, N, P \in {}_A\mathrm{FMod}$
- Let $P \subseteq M$. Then P is called a submodule of M.
- The quotient module M/P in ${}_A\mathrm{Mod}$ also belongs to ${}_A\mathrm{FMod}$.
- For a morphism $f : M \to N$ in ${}_A\mathrm{FMod}$, put $\mathrm{Ker}(f) := \{m \in M \mid f(m) = 0\}$ and $\mathrm{Im}(f) := \{f(m) \mid m \in M\}$. Then $\mathrm{Ker}(f)$ is a submodule of M in ${}_A\mathrm{FMod}$ and $\mathrm{Im}(f)$ is a submodule of N in ${}_A\mathrm{FMod}$.

Proposition

Let A be left s-unital. Suppose that $f : M \to N$ is a morphism in ${}_A\mathrm{FMod}$.

(a) *f is a monomorphism $\iff$ f is injective;*

(b) *f is an epimorphism $\iff$ f is surjective;*

(c) *f is an isomorphism $\iff$ f is bijective.*

Warning (Gonzales-Ferez and Marin)

If A is not s-unital, then monomorphisms in ${}_A\mathrm{FMod}$ may be non-injective!

Definition

- Let M be a module in ${}_A\mathrm{FMod}$.
- We say that M is simple in ${}_A\mathrm{FMod}$ if M is nonzero, and M has no submodules in ${}_A\mathrm{FMod}$ except the zero module and itself.
- We say that M is semisimple in ${}_A\mathrm{FMod}$ if M is the direct sum of simple submodules in ${}_A\mathrm{FMod}$.
- We say that A is left semisimple if A is semisimple in ${}_A\mathrm{FMod}$.

Theorem B

Suppose that A is a left s-unital ring. Then A is left semisimple if and only if every module in ${}_A\mathrm{FMod}$ is semisimple.

Definition

- Let $f : B \to A$ be a ring homomorphism (A, B associative, not necessarily unital).
- In that case we say that A/B is a ring extension.
- We say that f is left unital (respectively s-unital, firm, unitary) if the left B-module $\mathrm{Res}_f(A)$ is unital (respectively s-unital, firm, unitary).
- Consider the restriction functor $\mathrm{Res}_f : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$.

Proposition

Let $f : B \to A$ be a ring homomorphism.

(a) f left unital $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{UMod} \to {}_B\mathrm{Mod}^1$;

(b) f left s-unital $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{UMod} \to {}_B\mathrm{SMod}$;

(c) f left unitary $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{UMod} \to {}_B\mathrm{UMod}$;

(d) f left firm $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{FMod} \to {}_B\mathrm{FMod}$.

Definition

- Let $f : B \to A$ be a ring homomorphism (A, B associative, not necessarily unital).
- A/B is called separable if the multiplication map $\mu : A \otimes_B A \to A$, defined by $\mu(a \otimes a') = aa'$, for $a, a' \in A$, has a section in the category of A-bimodules (that is there is an A-bimodule map $\sigma : A \to A \otimes_B A$ such that $\mu \circ \sigma = \mathrm{id}_A$).
- We say that f is left unital (respectively s-unital, firm, unitary) if the left B-module $\mathrm{Res}_f(A)$ is unital (respectively s-unital, firm, unitary).

Theorem C

Let A and B be left s-unital rings with B left semisimple. Suppose that $f : B \to A$ is a left s-unital ring homomorphism such that A/B is separable. Then A is left semisimple. (In other words, under these hypotheses, the functor Res_f reflects semisimple modules)

Theorem D

Let G be finite group of order $|G|$. Let B be a ring with local units such that $|G|B = B$. Put $A := B[G]$. Then A/B is separable. (here $f : B \to A$ is the inclusion)

Proof

- For $a \in A$, choose an idempotent $e \in B$ with $ae = a = ea$.
- Put $n := |G|$. Since $nB = B$, the element ne is invertible in the unital ring eBe.
- Define
$$\sigma(a) := \sum_{g \in G} ag \otimes (ne)^{-1} g^{-1}.$$
- Show $\mu_{A/B} \circ \sigma = \mathrm{id}_A$ and σ is an A-bimodule map. □

Question

In Theorem C, why does Res_f reflect semisimple modules? Res_f is a separable functor!

Definition (Năstăsescu, Van den Bergh and Van Oystaeyen 1989)

A functor $F : C \to D$ is called *separable* if for all $M, N \in C_0 := Ob(C)$, there is a map

(S0) $R_{M,N} : \mathrm{Hom}_D(F(M), F(N)) \to \mathrm{Hom}_C(M, N)$ such that

(S1) $R_{M,N}(F(f)) = f$, for $f : M \to N$ in C, and

(S2)
$$F(g)f' = g'F(f) \quad \Longrightarrow \quad gR_{M,N}(f') = R_{M',N'}(g')f,$$
for $f : M \to M'$, $g : N \to N'$ in C, $f' : F(M) \to F(N)$, $g' : F(M') \to F(N')$ in D.

Proposition (Rafael 1990)

A functor $F : C \to D$ is separable if and only if for all $M, N \in C_0$, there is a map $R_{M,N}$ satisfying (S0), (S1), and

(S3) $R_{M,P}(gf) = R_{N,P}(g)R_{M,N}(f)$ *for* $M, N, P \in C_0$, $f : F(M) \to F(N)$, $g : F(N) \to F(P)$, *whenever* $f \in F(\mathrm{Hom}_C(M, N))$ *or* $g \in F(\mathrm{Hom}_C(N, P))$.

Theorem E

Let A be a firm ring and let $f : B \to A$ be a left firm ring homomorphism. Then A/B is separable if and only if the restriction functor $\mathrm{Res}_f : {}_A\mathrm{FMod} \to {}_B\mathrm{FMod}$ is separable.

Proposition (Folklore)

Separable functor reflect...

- *limits, colimits, monomorphisms, epimorphisms, initial objects, terminal objects, zero objects.*
- *simple and semisimple objects (abelian categories)*

Questions

- What do we mean by a simple object in a general category?
- What do we mean by a semisimple object in a general category?
- Do separable functors reflect such properties?

Definition

- Let $f : M \to N$ be a morphism in a category C.
- f is called a monomorphism if $\forall g_1, g_2 : P \to M$ $fg_1 = fg_2 \Rightarrow g_1 = g_2$
- Consider the class of all monomorphisms in C with codomain M.
- We define an equivalence relation $\sim_M$ on this class as follows. Given monomorphisms $f : P \to M$ and $g : Q \to M$, we write $f \sim_M g$ if there exists an isomorphism $h : P \to Q$ with $f = gh$.
- The equivalence class $[f]_{\sim_M}$ of such a monomorphism $f : P \to M$ is called a subobject of M. We denote the collection of subobjects of M by $Sub_C(M)$.

Definition

We say that $M \in C_0$ is subobject trivial if $|Sub_C(M)| = 1$, subobject nontrivial if $|Sub_C(M)| \geq 2$, and subobject simple if $|Sub_C(M)| = 2$.

Proposition

Let $F : C \to D$ be a separable functor that preserves monomorphisms. Let $M \in C_0$ be subobject nontrivial and $F(M)$ subobject simple in D. Then M is subobject simple in C.

Definition

Suppose that $M \in C_0$. We say that M is subobject semisimple if every monomorphism $m : P \to M$ has a retraction $r : M \to P$.

Proposition

Suppose that $F : C \to D$ is a separable functor which preserves monomorphisms. Let $M \in C_0$ have the property that $F(M)$ is subobject semisimple in D. Then M is subobject semisimple in C.

Example

- Let *Set* denote the category of sets.
- In *Set*, the monomorphisms are precisely the injective maps.
- All sets are subobject semisimple.
- The singleton sets $\{*\}$ are the subobject simple objects.
- In *Set*, every subobject simple object is trivially subobject semisimple.
- In an arbitrary category, a subobject simple object need not be subobject semisimple. Indeed, consider the category whose objects are M and N, and whose only morphisms are $\mathrm{id}_M, \mathrm{id}_N$ and $f : N \to M$. Subobjects of M correspond to the monomorphisms into M, that is id_M and f. Therefore, M is subobject simple. But f does not have a retraction. Hence, M is not subobject semisimple.

Now we dualize!

Definition

- Let $f : M \to N$ be a morphism in a category C.
- f is called an epimorphism if $\forall h_1, h_2 : N \to Q \; h_1 f = h_2 f \Rightarrow h_1 = h_2$
- Consider the class of all epimorphisms in C with domain M.
- We define an equivalence relation $\approx_M$ on this class as follows. Given epimorphisms $f : M \to P$ and $g : M \to Q$, we write $f \approx_M g$ if there exists an isomorphism $h : Q \to P$ with $f = hg$.
- The equivalence class $[f]_{\approx_M}$ of such an epimorphism $f : M \to P$ is called a quotient object of M. We denote the collection of quotient objects of M by $Quot_C(M)$.

Definition

We say that an object $M \in C_0$ is quotient trivial if $|Quot_C(M)| = 1$, quotient nontrivial if $|Quot_C(M)| \geq 2$, and quotient simple if $|Quot_C(M)| = 2$.

Proposition

Let $F : C \to D$ be a separable functor that preserves epimorphisms. Let $M \in C_0$ be quotient nontrivial and $F(M)$ quotient simple in D. Then M is quotient simple in C.

Definition

Suppose that $M \in C_0$. We say that M is quotient semisimple if every epimorphism $e : M \to P$ has a section $s : P \to M$.

Proposition

Suppose that $F : C \to D$ is a separable functor which preserves epimorphisms. Let $M \in C_0$ have the property that $F(M)$ is quotient semisimple in D. Then M is quotient semisimple in C.

Example

- Let *Set* denote the category of sets.
- In *Set*, the epimorphisms are precisely the surjective maps.
- All sets are quotient semisimple.
- The two-element sets are the quotient simple objects.
- In *Set*, every quotient simple object is trivially quotient semisimple.
- By taking the dual of the category in the previous example, we see that for arbitrary categories, a quotient simple object is not necessarily quotient semisimple.

Question

Under what conditions on a category does subobject (quotient) simplicity imply subobject (quotient) semisimplicity?

Definition

- $I \in C_0$ is called initial if $|\mathrm{Hom}_C(I, M)| = 1$ for all $M \in C_0$.
- $T \in C_0$ is called terminal if $|\mathrm{Hom}_C(M, T)| = 1$, for all $M \in C_0$.
- An object that is both initial and terminal is called a zero object.
- A category with a zero object is called pointed.

Proposition

Suppose that C is a pointed category. Let $M \in C_0$.

(a) *M subobject simple $\implies$ M subobject semisimple;*

(b) *M quotient simple $\implies$ M quotient semisimple.*

Remark

- *Grp*, the category of groups, is pointed having the trivial group as a zero object.
- All standard categories of modules are pointed (if the zero module is included).
- *Set* is not a pointed category.

Example

- In *Grp*, monomorphisms are exactly injective homomorphisms.
- Let G denote a group.
- Two monomorphisms into G represent the same subobject of G if and only if they have the same image in G.
- Therefore, the elements of $Sub_{Grp}(G)$ correspond to the set of subgroups of G.
- By Cauchy's theorem G is subobject simple precisely when G is a cyclic group of prime order.
- By a result by Baer, G is subobject semisimple if and only if G is an abelian group all of whose elements have finite square-free order.

Example

- In *Grp*, epimorphisms are precisely surjective homomorphisms.
- Let G denote a group.
- The quotient object of G represented by an epimorphism $q : G \to Q$ corresponds to the quotient group $G/\ker(q)$.
- Quotient objects of G correspond to isomorphism classes of quotient groups of G.
- Hence G is quotient simple in the categorical sense if and only if G has no nontrivial proper normal subgroups. Thus, quotient simple groups are exactly the simple groups in the classical sense.
- One can show that G is quotient semisimple if and only if every normal subgroup of G is complemented (so-called *nc*-groups).
- No complete classification of *nc*-groups is known.

Proposition

Suppose that A is left s-unital. Let $M \in {}_A\mathrm{FMod}$.

(a) *The following assertions are equivalent:*
- *M is simple in ${}_A\mathrm{FMod}$;*
- *M is subobject simple in ${}_A\mathrm{FMod}$;*
- *M is quotient simple in ${}_A\mathrm{FMod}$.*

(b) *The following assertions are equivalent:*
- *M is semisimple in ${}_A\mathrm{FMod}$;*
- *M is quotient semisimple in ${}_A\mathrm{FMod}$;*
- *M is subobject semisimple in ${}_A\mathrm{FMod}$.*

Question

If A is s-unital, then ${}_A\mathrm{FMod}$ is not always a semisimple category (every semisimple is a direct sum of finitely many simples). But what is it then? Locally semisimple?

Thank you for your attention!