Hilbert's Basis Theorem for *Wildebeests*

The 8th SNAG Meeting, Linköping University, 2026

May 28, 2026

Per Bäck

Mälardalen University

This talk is mainly based on joint work with Aryapoor (2025a,b).
Additional results are with Richter (2023, 2024).

Convention. All rings in this talk are unital.

Background and motivation

Hilbert's basis theorem (1890)—central to commutative algebra.

Hilbert's twist (1903)—rings with *twisted* multiplication.

Noncommutative polynomial rings: *Ore extensions* (Ore, 1933).

Hilbert's basis theorem for Ore extensions
(Noether–Schmeidler, 1920; Cohn, 1971).

Background and motivation

Nonassociative Ore extensions (Nystedt–Öinert–Richter, 2018),
with Hilbert's basis theorem (B.–Richter, 2023, 2024)
and "flipped" variants (Aryapoor–B., 2025a).

What underpins Hilbert's basis theorem? (Aryapoor–B., 2025b)

One ring to rule them all?

There is such a ring—a *wildebeest*.

Outline

I. Polynomial rings:
the noncommutative case

II. Polynomial rings:
the nonassociative case

I. Polynomial rings:

the noncommutative case

I. Polynomial rings: *the noncommutative case*

Ore (1933) began the study of noncommutative polynomial rings:

"[...] the multiplication of polynomials is so restricted that the degree of a product is equal to the sum of the degrees of the factors."

We want a new multiplication on $R[X]$. View $R[X]$ as a left R-module,

$$\deg(pq) \leqslant \deg p + \deg q, \quad p, q \in S.$$

$$Xr = \sigma(r)X + \delta(r), \quad r, \sigma(r), \delta(r) \in R. \tag{1}$$

From associativity: σ an endomorphism, δ a σ-*derivation*:

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s.$$

A unique ring $R[X; \sigma, \delta]$—*the (left) generalized polynomial ring*:

$$(rX^m)(sX^n) = \sum_{i \in \mathbb{N}} (r\pi_i^m(s))\, X^{i+n},$$

$\pi_i^m \colon R \to R$, the sum of all words in σ, δ containing i σ's and $m - i$ δ's.

I. Polynomial rings: *the noncommutative case*

Definition (Ore extension)

(S, x) is a *(left) Ore extension of R* if:

(O1) S is a ring extension of R and $x \in S$;

(O2) S is associative;

(O3) S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(O4) $xR \subseteq Rx + R$.

Proposition

The following hold:

(i) $(R[X; \sigma, \delta], X)$ is an Ore extension of R.

(ii) Every Ore extension of R is isomorphic to some $R[X; \sigma, \delta]$.

I. Polynomial rings: *the noncommutative case*

In $R[X;\sigma,\delta]$, $Xr = \sigma(r)X + \delta(r)$ for any $r \in R$.

Example

$R[X] = R[X;\mathrm{id}_R, 0_R]$.

If $\sigma = \mathrm{id}_R$, a *differential polynomial ring* $R[X;\mathrm{id}_R,\delta]$: $Xr = rX + \delta(r)$.

Example

The *Weyl algebra* is $K\langle X, Y\rangle/(XY - YX - 1) \cong K[Y][X;\mathrm{id}_{K[Y]}, \mathrm{d}/\mathrm{d}Y]$.
In quantum physics, X and Y are position and momentum operators.

If $\delta = 0_R$, a *skew polynomial ring* $R[X;\sigma,0_R] =: R[X;\sigma]$: $Xr = \sigma(r)X$.

Example

Let $*\colon \mathbb{C} \to \mathbb{C}, u \mapsto u^*$ be complex conjugation. In $\mathbb{C}[X;*]$, $Xu = u^*X$.
We have $\mathbb{C} \cong \mathbb{R}[X]/(X^2+1)$ and $\mathbb{H} \cong \mathbb{C}[X;*]/(X^2+1)$.

$\mathbb{H}$ is an $\mathbb{R}$-algebra with basis $\{1, i, j, k\}$ and $i^2 = j^2 = k^2 = ijk = -1$.

I. Polynomial rings: *the noncommutative case*

Definition (Noetherian ring)

R is *left (right) Noetherian* if any of these equivalent conditions hold:

(i) Any ascending chain of left (right) ideals stabilizes.

(ii) Any left (right) ideal is finitely generated.

(iii) Any nonempty set of left (right) ideals has a maximal element.

Theorem (Noether–Schmeidler, 1920; Cohn, 1971)

Let σ be an automorphism of R. If R is left (right) Noetherian, then so is $R[X; \sigma, \delta]$.

II. Polynomial rings:

the nonassociative case

II. Polynomial rings: *the nonassociative case*

R a **nonassociative ring** w. **additive maps σ, δ,** $\sigma(1) = 1$, $\delta(1) = 0$.

The *(left) generalized polynomial ring* $R[X;\sigma,\delta]$— $R[X]$ with

$$(rX^m)(sX^n) = \sum_{i\in\mathbb{N}} (r\pi_i^m(s))\,X^{i+n}, \quad \text{for any } r,s\in R,\ m,n\in\mathbb{N}.$$

Definition (Nonassociative Ore extension)

(S,x) is a ***nonassociative*** *Ore extension (**NOE**) of* R if:

(N1) S is a ring extension of R and $x\in S$;

(N2) $\boldsymbol{x \in N_m(S) \cap N_r(S)}$;

(N3) S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(N4) $xR \subseteq Rx + R$.

Theorem (B.–Richter, 2024)

Let σ be surjective. If R is right Noetherian, then so is $R[X;\sigma,\delta]$.

There is a bijection σ and a δ s.t. $K[Y][X;\sigma,\delta]$ is *not* left Noetherian.

II. Polynomial rings: *the nonassociative case*

With the *flip map* $\tau\colon R \times R \to R \times R$, $\tau(r, s) := (s, r)$, we can "flip" the multiplication on $R[X; \sigma, \delta]$ to get $R[X; \sigma, \delta]^{\mathrm{fl}}$:

$$(rX^m)(sX^n) = \sum_{i \in \mathbb{N}} \tau_n(r, \pi_i^m(s))X^{i+n}.$$

Here, τ_n is composition of the multiplication on R and τ^n, i.e.

$$\tau_n(r, s) = \begin{cases} rs & \text{if } n \text{ is even,} \\ sr & \text{if } n \text{ is odd.} \end{cases}$$

> ***Example***
> In $R[X]^{\mathrm{fl}}$, $(rX^m)(sX^n) = \tau_n(r, s)X^{m+n}$, so $r(sX) = (rX^0)(sX^1) = (sr)X$.

II. Polynomial rings: *the nonassociative case*

A $*$-*algebra* A is a nonassociative K-algebra with an *involution* $*\colon A \to A, a \mapsto a^*$ (i.e. $(ab)^* = b^* a^*$, $(a^*)^* = a$ for any $a, b \in A$).

$\mathrm{Cay}(A, \mu)$ with $\mu \in K \setminus \{0\}$ is $A \oplus A$ where for any $a, b, c, d \in A$,

$$(a, b)(c, d) := (ac + \mu d^* b, da + bc^*),$$
$$(a, b)^* := (a^*, -b).$$

Example

Start with $* = \mathrm{id}_K$ on $K = \mathbb{R}$, choose $\mu = \pm 1$ and then double:

$$\mathrm{Cay}(\mathbb{R}, -1) \cong \mathbb{C}, \qquad \mathrm{Cay}(\mathbb{R}, +1) \cong \mathbb{C}'$$
$$\mathrm{Cay}(\mathbb{C}, -1) \cong \mathbb{H}, \qquad \mathrm{Cay}(\mathbb{C}', +1) \cong \mathbb{H}'$$
$$\mathrm{Cay}(\mathbb{H}, -1) \cong \mathbb{O}, \qquad \mathrm{Cay}(\mathbb{H}', +1) \cong \mathbb{O}',$$
$$\vdots \qquad\qquad \vdots$$

II. Polynomial rings: *the nonassociative case*

Any nonassociative $*$-algebra A gives a FNOE $A[X;*]^{\mathsf{fl}}$. We extend $*$:

$$(a_0 + a_1X + a_2X^2 + \cdots)^* := a_0^* - a_1X + a_2^*X^2 - \cdots, \quad a_0, a_1, a_2, \ldots \in A.$$

Theorem (Aryapoor–B., 2025a)

$\mathrm{Cay}(A, \mu)$ and $A[X;*]^{\mathsf{fl}}/(X^2 - \mu)$ are isomorphic as $*$-algebras.

The flipped side of the moon:

$$\mathbb{C} \cong \mathbb{R}[X]/(X^2+1), \qquad \mathbb{C}' \cong \mathbb{R}[X]/(X^2-1),$$
$$\mathbb{H} \cong \mathbb{C}[X;*]/(X^2+1), \qquad \mathbb{H}' \cong \mathbb{C}'[X;*]/(X^2-1),$$
$$\mathbb{O} \cong \mathbb{H}[X;*]^{\mathsf{fl}}/(X^2+1), \qquad \mathbb{O}' \cong \mathbb{H}'[X;*]^{\mathsf{fl}}/(X^2-1),$$
$$\vdots \qquad\qquad \vdots$$

Definition (Generalized nonassociative Ore extension)

(S, x) is a *(left)* ***generalized*** *nonassociative Ore extension* (**GNOE**) *of* R if:

(G1) S is a ring extension of R and $x \in S$;

(G2) x **is power-associative**;

(G3) S is a free left R-module with basis $\{1, x, x^2, \ldots\}$;

(G4) $(\mathbf{Rx^m})(\mathbf{Rx^n}) \subseteq \mathbf{R} + \mathbf{Rx} + \cdots + \mathbf{Rx^{m+n}}$ **for all** $\boldsymbol{m, n} \in \mathbb{N}$.

Remark

(G4) is equivalent to $\deg(pq) \leqslant \deg p + \deg q$ for any $p, q \in S$.

Proposition (Aryapoor–B., 2025b)

Any FNOE of R is a GNOE of R. Also, if (S, x) is a GNOE of R, then:

(i) (S, x) is an Ore extension of R if and only if S is associative.

(ii) (S, x) is a NOE of R if and only if $x \in N_m(S) \cap N_r(S)$.

II. Polynomial rings: *the nonassociative case*

A *gnoe* or *wildebeest*.

Definition (Euclidean division algorithm)

A GNOE (S, x) of R satisfies the *(left) Euclidean division algorithm* if:

(D) For all $p_1, \ldots, p_n \in S$ and nonzero $q \in S$, if $\deg q \geqslant \max_i \deg p_i$ and $\mathrm{lc}(q) \in \langle \mathrm{lc}(p_1), \ldots, \mathrm{lc}(p_n) \rangle_R$, then there is $s \in \langle p_1, \ldots, p_n \rangle_S^{\text{deg-add}}$ s.t. $\deg(q - s) < \deg q$.

Theorem (Aryapoor–B., 2025b)

Let (S, x) be a GNOE of R that satisfies the Euclidean division algorithm. If R is right Noetherian, then so is S.

Proposition (Aryapoor–B., 2025b)

If σ is surjective, then $R[X; \sigma, \delta]$ and $R[X; \sigma, \delta]^{\text{fl}}$ satisfy the Euclidean division algorithm.

II. Polynomial rings: *the nonassociative case*

What about *right* GNOES?

Theorem (Aryapoor–B., 2025b)

Let (S, X) be a *right* GNOE of R that satisfies the *right* Euclidean division algorithm. If R is *left* Noetherian, then so is S.

Proposition (Aryapoor–B., 2025b)

$R[X; \sigma, \delta]$, $R[X; \sigma, \delta]^{\mathrm{fl}}$ are *right* GNOES of R if and only if σ is bijective.

Proposition (Aryapoor–B., 2025b)

If σ is an automorphism of R, then $R[X; \sigma, \delta]$ satisfies the *right* Euclidean division algorithm.

Example

If R is *not right* Noetherian, then $R[X]^{\mathrm{fl}}$ is *not left* Noetherian.

Q: Are there other examples of GNOEs?

References I

[AB25a] M. Aryapoor and P. Bäck. “Flipped Non-Associative Polynomial Rings and the Cayley–Dickson Construction”. In: *J. Algebra* 662 (2025).

[AB25b] M. Aryapoor and P. Bäck. “Hilbert’s Basis Theorem for Generalized Nonassociative Ore Extensions”. In: arXiv: 2512.02597 (2025).

[BR23] P. Bäck and J. Richter. “Hilbert’s Basis Theorem for Non-Associative and Hom-Associative Ore Extensions”. In: *Algebr. Represent. Theory* (2023).

[BR24] P. Bäck and J. Richter. “Non-Associative Versions of Hilbert’s Basis Theorem”. In: *Colloq. Math.* 176.1 (2024).

[Coh71] P. M. Cohn. *Free Rings and Their Relations*. London and New York: Academic Press, 1971.

References II

[Hil03] D. Hilbert. *Grundlagen der Geometrie*. Leipzig: Teubner, 1903.

[Hil90] D. Hilbert. “Über die Theorie der Algebraischen Formen”. In: *Math. Ann.* 42 (1890).

[NÖR18] P. Nystedt, J. Öinert, and J. Richter. “Non-Associative Ore Extensions”. In: *Isr. J. Math.* 224.1 (2018).

[NS20] E. Noether and W. Schmeidler. “Moduln in Nichtkommutativen Bereichen, Insbesondere aus Differential- und Differenzenausdrücken”. In: *Math. Z.* 8 (1920).

[Ore33] O. Ore. “Theory of Non-Commutative Polynomials”. In: *Ann. Math* 34.3 (1933).

…and in the darkness bind them.