On n-ary Hom-algebra structures

AMS MSC 2020: **17B61 Hom-Lie and related algebras,**

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Hom-Nambu and Hom-Nambu-Lie n-ary algebras

H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n-ary Nambu Algebras and Beyond, Journal of Mathematical Physics, 50, 083501,2009

An *n-ary Hom-Nambu algebra* $(V, [\cdot, \dots, \cdot], \alpha)$ is
V is linear space, $\alpha = (\alpha_i : V \to V)_{i=1,\cdots,n-1}$ are linear maps,
$[\cdot, \dots, \cdot] : V^{\times n} \to V$ is n-linear map (n-ary product) satisfying:

The n-ary Hom-Nambu identity

$\forall\ (x_1, \cdots, x_{2n-1}) \in V^{2n-1}$,

$$[\alpha_1(x_1), ..., \alpha_{n-1}(x_{n-1}), [x_n, ..., x_{2n-1}]] =$$

$$\sum_{i=n}^{2n-1} [\alpha_1(x_n), ..., \alpha_{i-n}(x_{i-1}), [x_1, \cdots, x_{n-1}, x_i], \alpha_{i-n+1}(x_{i+1}) ..., \alpha_{n-1}(x_{2n-1})]$$

n-Ary Hom-Nambu-Lie algebra is skew-symmetric n-ary Nambu algebra $\big(V, [\cdot, \dots, \cdot], (\alpha_1, \dots, \alpha_{n-1})\big)$, that is,
$\forall\, \sigma \in \mathcal{S}_n$ and $\forall\, x_1, \dots, x_n \in V$

$$[x_{\sigma(1)}, \dots, x_{\sigma(n)}] = Sgn(\sigma)[x_1, \dots, x_n]$$

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Ternary Hom-Nambu algebras

$$\begin{aligned}[\alpha_1(x_1), \alpha_2(x_2), [x_3, x_4, x_5]] = \\ [[x_1, x_2, x_3], \alpha_1(x_4), \alpha_2(x_5)] + [\alpha_1(x_3), [x_1, x_2, x_4], \alpha_2(x_5)] \\ + [\alpha_1(x_3), \alpha_2(x_4), [x_1, x_2, x_5]].\end{aligned}$$

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n-ary Hom-Nambu algebras

(V, m) is n-ary Nambu (or Nambu-Lie) algebra
$\rho : V \to V$ is n-ary Nambu (or Nambu-Lie) algebras endomorphism.
$m_\rho = \rho \circ m$
$\widetilde{\rho} = (\rho, \cdots, \rho)$.
Then $(V, m_\rho, \widetilde{\rho})$ is an n-ary Hom-Nambu (Hom-Nambu-Lie) algebra.

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Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Arnlind, J., Makhlouf, A., Silvestrov, S., Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras. J. Math. Phys. 51, no. 4, 043515, (2010)

Definition

$(V, [\cdot,\cdot])$ binary algebra (2-ary algebra), $\tau : V \to \mathbb{K}$ linear map.
Ternary bracket (trilinear map) $[\cdot,\cdot,\cdot]_\tau : V \times V \times V \to V$:

$$[x, y, z]_\tau = \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y].$$

If $[\cdot,\cdot]$ is skew-symmetric, then $[\cdot,\cdot,\cdot]_\tau$ is skew-symmetric.

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Hom-Nambu-Lie algebras induced from Hom-Lie algebras

If τ is a linear function such that $\tau([x, y]) = 0$ for all $x, y \in V$, then we call τ a *trace function on* $(V, [\cdot, \cdot])$. It follows immediately that $\tau([x, y, z]_\tau) = 0$ for all $x, y, z \in V$ if τ is a trace function.

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

Theorem

$(V, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and $\beta : V \to \mathbb{K}$ be a linear map. Assume that τ is a trace function on V fulfilling

$$\tau\big(\alpha(x)\big)\tau(y) = \tau(x)\tau\big(\alpha(y)\big)$$
$$\tau\big(\beta(x)\big)\tau(y) = \tau(x)\tau\big(\beta(y)\big)$$
$$\tau\big(\alpha(x)\big)\beta(y) = \tau\big(\beta(x)\big)\alpha(y)$$

for all $x, y \in V$.

Then $(V, [\cdot,\cdot,\cdot]_\tau, (\alpha, \beta))$ is a Hom-Nambu-Lie algebra. If we choose $\beta = \alpha$ conditions for trace τ reduce to

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)).$$

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Example

V vector space of $n \times n$ matrices
$\alpha(x) = s^{-1}xs$ for an invertible matrix s
Then $(V, \alpha \circ [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra. For matrices, any trace function is proportional to the matrix trace, so we let $\tau(x) = \mathrm{tr}(x)$. If we want to choose a $\beta \neq 0$, it can be proved that β has to be proportional to α, i.e. $\beta = \lambda\alpha$ for some $\lambda \neq 0$. Since $\mathrm{tr}\left(\alpha(x)\right) = \mathrm{tr}(x)$ it is clear that $(\alpha, \lambda\alpha, \mathrm{tr})$ is a nondegenerate compatible triple on V, which implies that $\left(V, [\cdot, \cdot, \cdot]_{\mathrm{tr}}, (\alpha, \lambda\alpha)\right)$ is a Hom-Nambu-Lie algebra induced from $(V, \alpha \circ [\cdot, \cdot], \alpha)$.

Some key references on n-ary Hom-algebra structures (cont.)

- Ataguema, H., Makhlouf, A., Silvestrov, S., Generalization of n-ary Nambu Algebras and Beyond, J. Math. Phys., 50, 083501, (2009)
- Arnlind, J., Makhlouf, A., Silvestrov, S., Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras. J. Math. Phys. 51, no. 4, 043515, (2010)
- Arnlind, J., Makhlouf, A., Silvestrov, S., Construction of n-Lie algebras and n-ary Hom-Nambu-Lie algebras, J. Math. Phys. 52,, no. 12, 123502, (2011)
- Arnlind, J., Kitouni, A., Makhlouf, A., Silvestrov, S., Structure and cohomology of 3-Lie algebras induced by Lie algebras. In: *Algebra, Geometry and Mathematical Physics*; Springer Proceedings in Mathematics and Statistics; Springer: Cham, Switzerland, Volume 85, p. 123–144 (2014)
- Kitouni, A., Makhlouf, A., Silvestrov, S., On $(n+1)$-Hom-Lie algebras induced by n-Hom-Lie algebras. *Georgian Math. J.* **2016**, *23*, 75–95.
- Kitouni, A., Makhlouf, A., Silvestrov, S., On solvability and nilpotency for n-Hom-Lie algebras and $(n+1)$-Hom-Lie algebras induced by n-Hom-Lie algebras. In: *Algebraic Structures and Applications*; Springer Proceedings in Mathematics and Statistics; Springer: Cham, Switzerland, Volume 317, 127–157 (2020)
- Kitouni, A., Makhlouf, A., Silvestrov, S., On n-ary generalization of BiHom-Lie algebras and BiHom-associative algebras. In: *Algebraic Structures and Applications*; Springer Proceedings in Mathematics and Statistics; Springer: Cham, Switzerland, Volume 317, 99-126 (2020)
- Kitouni, A., Silvestrov, S., On Classification of $(n+1)$-dimensional n-Hom-Lie algebras with nilpotent twisting maps. In: *Non-commutative and Non-associative Algebra and Analysis Structures*; Proceedings in Mathematics and Statistics; Springer: Berlin/Heidelberg, Germany, Volume 426, 525–562 (2023)
- Kitouni, A., Silvestrov, S., On properties and classification of a class of 4-dimensional 3-Hom-Lie Algebras with a Nilpotent Twisting Map, Axioms, 13, 373. 44 pp, (2024) `https://doi.org/10.3390/axioms13060373`

4-dimensional 3-Hom-Lie algebra $4_{3,N(2),6}$

- Kitouni, A., Silvestrov, S., On Classification of $(n+1)$-dimensional n-Hom-Lie algebras with nilpotent twisting maps. In *Non-commutative and Non-associative Algebra and Analysis Structures*; Silvestrov, S., Malyarenko A., Eds.; Proceedings in Mathematics and Statistics; Springer: Berlin/Heidelberg, Germany, Volume 426, 525–562 (2023)
- Kitouni, A., Silvestrov, S., On properties and classification of a class of 4-dimensional 3-Hom-Lie Algebras with a Nilpotent Twisting Map, Axioms, 13, 373. 44 pp, (2024) `https://doi.org/10.3390/axioms13060373`

Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an n-ary skew-symmetric algebra of dimension $n+1$ with a linear map α. Given a basis $\{e_i\}_{1\leq i\leq n+1}$ of A as linear space, the linear map α is fully determined by its matrix determined by action of α on the basis, and a skew-symmetric n-ary multi-linear operation (bracket) is fully determined by $[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}]$ for all $1 \leq i \leq n+1$, represented by a matrix $B = (b(i,j))_{1\leq i,j\leq n+1}$, as follows:

$$[e_1, \ldots, \widehat{e_i}, \ldots, e_{n+1}] = (-1)^{n+1+i} w_i, \quad (1)$$

$$w_i = \sum_{p=1}^{n+1} b(p,i) e_p, \quad (w_1, \ldots, w_{n+1}) = (e_1, \ldots, e_{n+1}) B.$$

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4-dimensional 3-Hom-Lie algebra $4_{3,N(2),6}$, cont.

An interesting class of 4-dimensional 3-Hom-Lie algebras $4_{3,N(2),6}$ is defined according to (1) on the basis $\{e_i\}_{1\leq i\leq 4}$ by

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & c(1,3,4,1) & -c(1,2,4,1) & 0 \\ 0 & c(1,3,4,2) & -c(1,2,4,2) & 0 \\ 0 & c(1,3,4,3) & -c(1,2,4,3) & 0 \\ 0 & c(1,3,4,4) & -c(1,2,4,4) & 0 \end{pmatrix},$$

$$\begin{aligned}
[e_1, e_2, e_3] &= 0 \\
[e_1, e_2, e_4] &= c(1,2,4,1)e_1 + c(1,2,4,2)e_2 + c(1,2,4,3)e_3 + c(1,2,4,4)e_4 \\
[e_1, e_3, e_4] &= c(1,3,4,1)e_1 + c(1,3,4,2)e_2 + c(1,3,4,3)e_3 + c(1,3,4,4)e_4 \\
[e_2, e_3, e_4] &= 0,
\end{aligned}$$

where $c(i_1, \ldots, i_n, k) = c^k_{i_1,\ldots,i_n}$ are the structure constants as in

$$[e_{i_1}, \ldots, e_{i_n}] = \sum_{k=1}^{\dim A} c^k_{i_1,\ldots,i_n} e_k = \sum_{k=1}^{\dim A} c(i_1, \ldots, i_n, k) e_k.$$

The 3-Hom-Lie algebra from $4_{3,N(2),6}$ is multiplicative if and only if

$$c(1,2,4,3) = 0,\ c(1,2,4,4) = 0,\ c(1,3,4,3) = 0,\ c(1,3,4,4) = 0.$$

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Derived and Central Descending Series of *n*-ary Hom-algebras

Kitouni, A., Mboya, S., Ongong'a, E., Silvestrov, S. (2023). On Ideals and Derived and Central Descending Series of n-ary Hom-Algebras. In: Non-Associative Algebras and Related Topics. NAART 2020. Springer Proceedings in Mathematics and Statistics, vol 427. Springer, Cham.
https://doi.org/10.1007/978-3-031-32707-0_17

The aim of this work is to explore some properties of *n*-ary skew-symmetric Hom-algebras and *n*-Hom-Lie algebras related to their ideals, derived series and central descending series. We extend the notions of derived series and central descending series to *n*-ary skew-symmetric Hom-algebras and provide various general conditions for their members to be Hom-subalgebras, weak ideals or Hom-ideals in the algebra or relatively to each other. In particular we study the invariance under the twisting maps of the derived series and central descending series and their subalgebra and ideal properties for a class of 3-dimensional Hom-Lie algebra and some 4-dimensional 3-Hom-Lie algebras. We also introduce a type of generalized ideals in *n*-ary Hom-algebras and present a few basic properties.

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Derived and Central Descending Series of n-ary Hom-algebras

If $\beta : \mathcal{A} \to \mathcal{A}$ is a weak morphism of n-Hom-Lie algebra $\mathcal{A} = (A, [\cdot, \dots, \cdot], \{\alpha_i\}_{1\le i\le n-1})$, and $[x_1, \dots, x_n]_\beta := \beta([x_1, \dots, x_n])$, then, $\left(A, [\cdot, \dots, \cdot]_\beta, \{\beta \circ \alpha_i\}_{1\le i\le n-1}\right)$ is an n-ary Hom-Lie algebra.
If the algebra $(A, [\cdot, \dots, \cdot], \alpha)$ is multiplicative and $\beta \circ \alpha = \alpha \circ \beta$, then $\left(A, [\cdot, \dots, \cdot]_\beta, \beta \circ \alpha\right)$ is multiplicative.

If $(A, [\cdot, \dots, \cdot])$ is an n-Lie algebra, $\beta : A \to A$ an algebra morphism, and $[x_1, \dots, x_n]_\beta = \beta([x_1, \dots, x_n])$, then, $\left(A, [\cdot, \dots, \cdot]_\beta, \beta\right)$ is a multiplicative n-Hom-Lie algebra.

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Derived and Central Descending Series of n-ary Hom-algebras

Let $(A, [\cdot, \dots, \cdot], \alpha_1, \dots, \alpha_{n-1})$ be an n-Hom-Lie algebra or more generally an n-ary skew-symmetric Hom-algebra, and let I be a weak ideal of A. For $2 \leq k \leq n$.

k-Derived series of the ideal I:

$$D_k^0(I) = I \text{ and } D_k^{p+1}(I) = \langle[\underbrace{D_k^p(I), \dots, D_k^p(I)}_{k}, \underbrace{A, \dots, A}_{n-k}]\rangle.$$

k-Central descending series of I:

$$C_k^0(I) = I \text{ and } C_k^{p+1}(I) = \langle[C_k^p(I), \underbrace{I, \dots, I}_{k-1}, \underbrace{A, \dots, A}_{n-k}]\rangle.$$

Derived and Central Descending Series of n-ary Hom-algebras

Let $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ be an n-Hom-Lie algebra or more generally an n-ary skew-symmetric Hom-algebra, and let I be a weak ideal of A. For $2 \leq k \leq n$, the ideal I is said to be k-solvable (resp. k-nilpotent) if there exists $r \in \mathbb{N}$ such that $D_k^r(I) = \{0\}$ (resp. $C_k^r(I) = \{0\}$), and smallest $r \in \mathbb{N}$ satisfying this condition is called the class of k-solvability (resp. the class of k-nilpotency) of I.

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Derived and Central Descending Series of n-ary Hom-algebras

Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot]_A, \alpha_1, \ldots, \alpha_{n-1})$, $\mathcal{B} = (B, [\cdot, \ldots, \cdot]_B, \beta_1, \ldots, \beta_{n-1})$ be two n-ary skew-symmetric Hom-algebras, $f : \mathcal{A} \to \mathcal{B}$ be a surjective n-ary Hom-algebras morphism and I a weak ideal of $\mathcal{A}$. Then, for all $r \in \mathbb{N}$ and $2 \leq k \leq n$,

$$f\left(D_k^r(I)\right) = D_k^r\left(f(I)\right) \text{ and } f\left(C_k^r(I)\right) = C_k^r\left(f(I)\right).$$

Remark: This statement implies that if two n-Hom-Lie algebras are isomorphic, they would also have isomorphic terms of the derived series and central descending series, which also means that if two algebras have a significant difference in the derived series or the central descending series, for example different dimensions of given corresponding members, then these algebras cannot be isomorphic.

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Derived and Central Descending Series of n-ary Hom-algebras

Let $(A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_{n-1})$ be an n-Hom-Lie algebra or more generally an n-ary skew-symmetric Hom-algebra. Define $Z(A)$, the center of A, by

$$Z(A) = \{z \in A : [x_1, \ldots, x_{n-1}, z] = 0, \forall x_1, \ldots, x_{n-1} \in A\}.$$

Note that $Z(A)$ is a weak ideal of $\mathcal{A}$.

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Derived and Central Descending Series of n-ary Hom-algebras

Let $(A, [\cdot, \ldots, \cdot], \alpha)$ be an n-Hom-Lie algebra. If A is k-nilpotent, for any $2 \leq k \leq n$, then the center $Z(A)$ of A is not trivial.
When $\mathcal{A} = (A, [\cdot, \ldots, \cdot], (\alpha_i)_{1 \leq i \leq n-1})$ is an n-ary skew-symmetric Hom-algebra, it holds that

- If $\mathcal{A}$ is nilpotent, then $Z(\mathcal{A})$ is not trivial.
- If $\dim A = n + 1$, then $\dim Z(\mathcal{A}) = 0$ or $\dim Z(\mathcal{A}) = 1$ or $Z(\mathcal{A}) = A$.

Derived and Central Descending Series of n-ary Hom-algebras

Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1 \leq i \leq n-1})$ be an $(n+1)$-dimensional n-ary skew-symmetric algebra. The algebra $\mathcal{A}$ is nilpotent and non abelian if and only if $\dim Z(\mathcal{A}) = 1$ and $[A, \ldots, A] = Z(\mathcal{A})$.

Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot], \{\alpha_i\}_{1 \leq i \leq n-1})$ be an n-Hom-Lie algebra or more generally an n-ary skew-symmetric Hom-algebra. $\mathcal{A}$ is nilpotent of class p if and only if $\{0\} \subsetneq C^{p-1}(A) \subseteq Z(A)$.

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Ideal Properties of Derived and Central Descending Series

Let $\mathcal{A} = (A, [\cdot, \ldots, \cdot], \alpha_1, \ldots, \alpha_s)$ be an n-ary skew-symmetric Hom-algebra, and let I be a weak ideal of $\mathcal{A}$. Then,

1. for $2 \leq k \leq n-1$ and $r \in \mathbb{N}$,

$$D_{k+1}^r(I) \subseteq D_k^r(I) \text{ and } C_{k+1}^r(I) \subseteq C_k^r(I);$$

2. for $2 \leq k \leq n$ and $r \in \mathbb{N}$

$$C_k^{r+1}(I) \subseteq C_k^r(I) \text{ and } D_k^{r+1}(I) \subseteq D_k^r(I).$$

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Ideal Properties of Derived and Central Descending Series

Let $\mathcal{A} = (A, [\cdot, \dots, \cdot], \alpha_1, \dots, \alpha_s)$ be an n-ary skew-symmetric Hom-algebra, and let I be a weak ideal of $\mathcal{A}$.
For all $2 \le k \le n$ and all $r \in \mathbb{N}$,

$D_k^r(I)$ and $C_k^r(I)$ are weak subalgebras of $\mathcal{A}$.

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Ideal Properties of Derived and Central Descending Series

Let $\mathcal{A} = (A, [\cdot, \dots, \cdot], \alpha_1, \dots, \alpha_s)$ be an n-ary skew-symmetric Hom-algebra, and let I be an Hom-ideal of $\mathcal{A}$. If all the linear maps $\alpha_i, 1 \leq i \leq s$ are weak morphisms, then the subspaces $D_k^r(I)$ and $C_k^r(I)$ are Hom-subalgebras of $\mathcal{A}$, for all $2 \leq k \leq n$ and all $r \in \mathbb{N}$. If, in addition, these maps are surjective, and $\mathcal{A}$ is an n-Hom-Lie algebra then $D_k^r(I)$ and $C_k^r(I)$ are Hom-ideals of $\mathcal{A}$.

If $\mathcal{A} = (A, [\cdot, \dots, \cdot], (\alpha_i)_{1 \leq i \leq s})$ is an n-ary skew-symmetric Hom-algebra and I is a weak ideal of $\mathcal{A}$, then for all $p \in \mathbb{N}$, $2 \leq k \leq n$, $D_k^{p+1}(I)$ is a weak ideal of $D_k^p(I)$, and $C_k^{p+1}(I)$ is a weak ideal of $C_k^p(I)$. In particular, $D_k^1(A)$ and $C_k^1(A)$ are weak ideals of $\mathcal{A}$.

Ideal Properties of Derived and Central Descending Series

Let $\mathcal{A} = (A, [\cdot, \dots, \cdot], (\alpha_i)_{1 \leq i \leq s})$ be an n-ary skew-symmetric Hom-algebra and let I be a Hom-ideal of $\mathcal{A}$. If for all $1 \leq i \leq s$, α_i satisfies one of the following conditions

1. $\forall x_1, \dots, x_n \in A$,
   $\alpha_i([x_1, \dots, x_n]) = [\alpha_i(x_1), \dots, \alpha_i(x_n)]$ (α_i is a weak morphism),
2. $\forall x_1, \dots, x_n \in A$, $\alpha_i([x_1, \dots, x_n]) = [x_1, \dots, x_{n-1}, \alpha_i(x_n)]$,
3. α_i are derivations : $\forall x_1, \dots, x_n \in A$,
   $$\alpha_i([x_1, \dots, x_n]) = \sum_{k=1}^{n} [x_1, \dots, x_{k-1}, \alpha_i(x_k), x_{k+1}, \dots, x_n]$$

then, $\forall p \in \mathbb{N}$ and $2 \leq k \leq n$, $D_k^p(I)$ and $C_k^p(I)$ are Hom-subalgebras of $\mathcal{A}$.

If I is a Hom-ideal of $\mathcal{A}$ and each $\alpha_i, 1 \leq i \leq s$ satisfies either condition 1, 2 or 3, then $D_k^{p+1}(I)$ is a Hom-ideal of $D_k^p(I)$ and $C_k^{p+1}(I)$ is a Hom-ideal of $C_k^p(I)$.

Let $\mathcal{A} = (A, [\cdot, \dots, \cdot], \alpha_1, \dots, \alpha_{n-1})$ be an n-ary Hom-Lie algebra and let I be a Hom-ideal of $\mathcal{A}$. If all the linear maps $\alpha_i, 1 \leq i \leq n-1$ are surjective, and each of them satisfies either condition 1, 2 or 3, then $D_k^r(I)$ and $C_k^r(I)$ are Hom-ideals of $\mathcal{A}$, for all $2 \leq k \leq n$ and all $r \in \mathbb{N}$.

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AB - ♥ BA = I

Thank you for the music,

the songs I'm singing!

Thanks for all the joy

they're bringing!