Involutions in the Cayley-Dickson construction

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Historical overview

Complex numbers

1545, G. Cardano	Idea of complex numbers (in *Ars Magna*)
1572, R. Bombelli	*L'Algebra*
1797, C. Wessel	Geometrical interpretation (in *Om directionens analytiske betegning*)
1837, Hamilton	Complex numbers as pairs of real numbers
$\rightarrow$	*Is there a 3-dimensional real division algebra?*

Hamilton’s bridge

$$i^2 = j^2 = k^2 = ijk = -1$$

Figure: The Quaternions bridge (formerly Broomebridge), Dublin

Historical overview

Complex numbers

1545, G. Cardano	Idea of complex numbers (in *Ars Magna*)
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1837, Hamilton	Complex numbers as pairs of real numbers

Quaternions and beyond

1843, Hamilton	Quaternions
J.T. Graves and A. Cayley	Octonions

Composition algebras

Definition

A *composition algebra* over a field k is a non-associative unital algebra with a norm N such that $N(xy) = N(x)N(y)$.

$\rightarrow$ *Norm:* $N : C \rightarrow k$

- $N(\lambda x) = \lambda^2 N(x), \quad \lambda \in k$
- $\langle \cdot, \cdot \rangle : C \times C \rightarrow k, \ \langle x, y \rangle = N(x + y) - N(x) - N(x)$ bilinear
- $C^\perp = \{x \in C \mid \langle x, y \rangle = 0, \forall y \in C\} = 0$

$\rightarrow$ *Conjugation:* $x^* := \langle x, 1 \rangle \cdot 1 - x$.

Definition

- *Composition division algebra*: $N(x) = 0 \implies x = 0$
- Otherwise, *split composition algebra*

Classification of composition algebras

Theorem (Hurwitz, 1898)

Dimension	*Associative*	*Commutative*	*Name*
1 ($\mathrm{char}(k) \neq 2$)	✓	✓	
2	✓	✓	
4	✓		*quaternion algebra*
8			*octonion algebra*

Theorem (see Springer-Veldkamp)

In dimension 2, 4, 8: one split composition algebra over a given field.

They are the only composition algebras containing zero divisors.

Cayley-Dickson construction

How to build composition algebras?

Definition (Cayley-Dickson double)

C: composition algebra
$\mu \in k \setminus \{0\}$
The *Cayley double* of C, written $\mathrm{Cay}(C, \mu)$, is defined by:

- underlying additive group: $C \oplus C$
- multiplication: $(x, y)(u, v) = (xu + \mu v^* y, yu^* + vx)$
- conjugation: $(x, y)^* = (x^*, -y)$
- $N(x, y) = N(x) - \mu N(y)$.

Example (Composition algebras over $\mathbb{R}$)

with $\mu = -1$: $\mathbb{C}, \mathbb{H}, \mathbb{O}$
with $\mu = 1$: $\mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s$

Some properties of Cayley-Dickson algebras

Proposition (see e.g. Springer-Veldkamp)

$\mathrm{Cay}(C, \mu)$ is a composition algebra if and only if C is associative.

Proposition (see e.g. Springer-Veldkamp)

$\mathrm{Cay}(C, \mu)$ is associative if and only if C is associative and commutative.

Proposition (see e.g. Springer-Veldkamp)

C composition algebra, D composition subalgebra
Then, $\mathrm{Cay}(D, \mu)$ is a composition subalgebra of C.

$*$-algebras

R: associative commutative unital ring
A: nonassociative algebra over R

Definition ($*$-algebra)

An *involution of* A is a R-linear map $* : A \to A$ that satisfies

- $(x^*)^* = x$
- $(xy)^* = y^*x^*$

$\forall x, y \in A$.
A nonassociative algebra equipped with an involution $*$ is called a *nonassociative* $*$*-algebra*.

Cayley double of a $*$-algebra

Definition (Cayley double)

$(A, *)$: $*$-algebra
$\mu \in R \setminus \{0\}$
The *Cayley double* of A, written $\mathrm{Cay}(A, \mu)$, is defined by:

- underlying additive group: $A \oplus CA$
- multiplication: $(x, y)(u, v) = (xu + \mu v^* y, yu^* + vx)$
- conjugation: $(x, y)^* = (x^*, -y)$.

Remark

$\mathrm{Cay}(A, \lambda^2 \mu) \cong \mathrm{Cay}(A, \mu)$

Question 1

Are there other choices for the involution in the Cayley-Dickson construction?

Involutions: the zero-divisor-free case

Corollary

A: $*$-algebra that has no zero divisors. Then, the only involutions of $\mathrm{Cay}(A, \mu)$ are the involutions α and β, defined by

$$\alpha(x, y) = (x^*, -y), \quad \beta(x, y) = (x^*, y).$$

Notation interlude

Definition

Left nucleus of A: $N_l(A) = \{a \in A \mid a(xy) = (ax)y, \forall x, y \in A\}$

Right nucleus $N_r(A)$, Middle nucleus $N_m(A)$, Nucleus $N(A)$.

Involutions: in general

Theorem

A: $*$-algebra.
A map $\gamma\colon \mathrm{Cay}(A,\mu) \to \mathrm{Cay}(A,\mu)$ is an involution of $\mathrm{Cay}(A,\mu)$ extending $*$ if and only if it is of the form

$$\gamma(x,y) = (x^* + ay^*, by) \qquad (x, y \in A),$$

for some $a, b \in A$ satisfying

$$a \in N(A) \quad \text{and} \quad (xa)^* = -xa \quad (x \in A), \tag{1}$$
$$b \in N_l(A) \quad \text{and} \quad b^2 = 1, \tag{2}$$
$$a = ba, \tag{3}$$
$$\mu xy = a^2(yx) + \mu(xb^*)(by), \quad (x, y \in A). \tag{4}$$

Involutions: an example in characteristic 2

Example

K: field, $\mathrm{char}(K) = 2$. Take $A = \mathbb{H}_s \cong M_2(K)$ with involution

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix}^* = \begin{pmatrix} t & -y \\ -z & x \end{pmatrix}$$

Lemma

$a \in N(A)$ and $(ax)^* = -ax \quad (x \in A) \implies [A, A]a = 0.$

Example (continuation)

$a = 0 \quad ; \quad b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Involutions: the 2-torsion-free case

Corollary

A: 2-torsion-free $*$-algebra.
A map $\gamma\colon \mathrm{Cay}(A,\mu) \to \mathrm{Cay}(A,\mu)$ is an involution of $\mathrm{Cay}(A,\mu)$ extending $*$ if and only if it is of the form

$$\gamma(x,y) = (x^* + ay^*, by) \qquad (x, y \in A),$$

for some $a, b \in A$ satisfying

$$a \in N(A) \quad \text{and} \quad a \text{ is proper skew,} \tag{5}$$
$$b \in N(A), \quad b^2 = 1, \quad \text{and} \quad b^* = b, \tag{6}$$
$$a = ba \tag{7}$$

$$\mu xy = a^2(yx) + \mu(xb^*)(by)$$

Involutions: the 2-torsion-free case

Example

A: nonassociative 2-torsion-free $*$-algebra over R
$a, b \in A$: as in the corollary
B: $*$-subalgebra of A generated by a and b

Then, B associative and

$$a^2 = 0, \quad a = ab, \quad ab = ba, \quad b^2 = 1, \quad a^* = -a, \quad b^* = b.$$

$\therefore B = K[a, b]/(a^2, a - ab, b^2 - 1)$, with involution: $a^* = -a$ and $b^* = b$

Scalar involutions

Definition (Scalar involution)

An involution is *scalar* if all norms xx^* are scalar.

Corollary

Let A be a nonassociative $*$-algebra. A map $\gamma\colon \mathrm{Cay}(A,\mu) \to \mathrm{Cay}(A,\mu)$ is a scalar involution of $\mathrm{Cay}(A,\mu)$ extending $*$ if and only if $*$ is scalar and

$$\gamma(x,y) = (x^*, -y) \qquad (x, y \in A).$$

An example: starting from $\mathbb{R}$

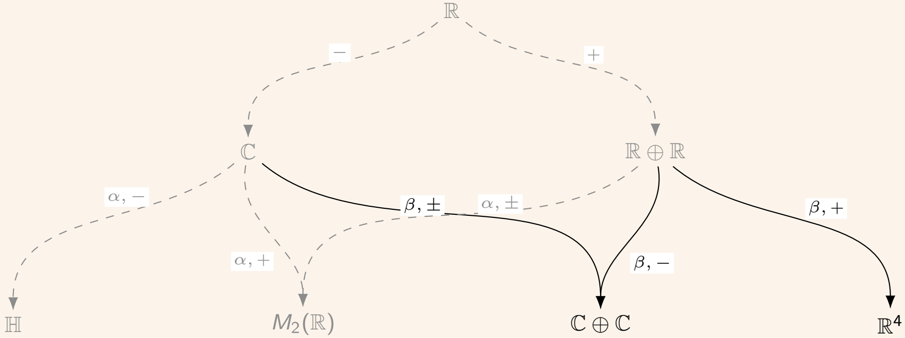


Figure: Cayley–Dickson doubling over $\mathbb{R}$. The edge labels $+$ and $-$ denote the sign of μ at each step. Dashed gray arrows indicate the classical (α-only) paths.

An example: starting from $\mathbb{R}$

n	$\dim_{\mathbb{R}} A_n$	A_n	α	β
0	1	$\mathbb{R}$		
1	2	$\mathbb{C}$	$\mathbb{C}$-conjugation	identity
		$\mathbb{R} \oplus \mathbb{R}$	exchange	identity
2	4	$\mathbb{H}$	$\mathbb{H}$-conjugation	$\mathbb{C}$-conjugation
		$M_2(\mathbb{R})$	adjugate	transpose
		$\mathbb{C} \oplus \mathbb{C}$	exchange; entrywise $\mathbb{C}$-conjugation	identity
		$\mathbb{R}^4$	block exchange	identity

Table: Cayley–Dickson doubling over $\mathbb{R}$.

Question 2

Which of the new involutions produce new $*$-algebras?

Algebra isomorphisms that preserve A

Theorem

A: $*$-algebra
Let $\phi : \mathsf{Cay}(A, \mu_1) \to \mathsf{Cay}(A, \mu_2)$, $\phi|_A = \mathrm{id}_A$.
It is an algebra isomorphism if

$$\phi(x, y) = (x + cy^*, dy)$$

for some $c, d \in A$ satisfying

$$c \in N(A) \quad \text{and} \quad (cx)^* = -cx \quad (x \in A), \tag{8}$$
$$d \in N_l(A) \quad \text{and} \quad x \mapsto dx \text{ is bijective}, \tag{9}$$
$$\mu_1 xy = c^2(xy) + \mu_2(xd^*)(dy) \quad (x, y \in A). \tag{10}$$

Algebra isomorphisms preserving A: more assumptions

Proposition

A: 2-torsion-free $*$-algebra, $\mu_1 = \mu_2$ cancellable
Let $\phi : \mathrm{Cay}(A, \mu_1) \to \mathrm{Cay}(A, \mu_2)$, $\phi|_A = \mathrm{id}_A$.
It is an algebra isomorphism if

$$\phi(x, y) = (x + cy^*, dy)$$

for some $c, d \in A$ satisfying

$$c \in N(A), \quad c \text{ is proper skew.} \tag{11}$$
$$d \in N(A), \quad dd^* = d^*d = 1, \tag{12}$$

for any $x, y \in A$.

$*$-algebra isomorphisms

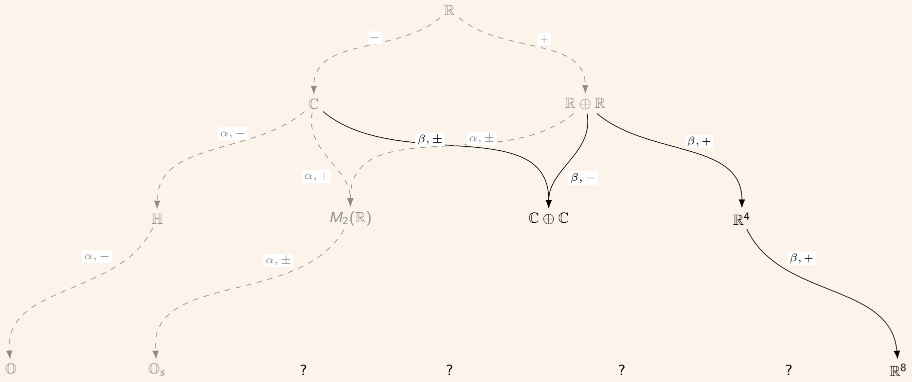
Corollary

γ_1, γ_2: involutions of $\mathrm{Cay}(A, \mu)$ defined by $\gamma_i(x, y) = (x^* + a_i y^*, b_i y)$.
Then, $\mathrm{Cay}_{\gamma_1}(A, \mu)$ and $\mathrm{Cay}_{\gamma_2}(A, \mu)$ are isomorphic as $*$-algebras $\iff$

$$a_1 = -c - b_1 c + a_2 d^* \quad \text{and} \quad d b_1 = b_2 d$$

for some $c, d \in A$, as previously.

Conclusion



$\mathbb{R}$
$-$
$+$
$\mathbb{C}$
$\mathbb{R} \oplus \mathbb{R}$
$\alpha, -$
$\beta, \pm$
$\alpha, \pm$
$\beta, +$
$\alpha, +$
$\beta, -$
$\mathbb{H}$
$M_2(\mathbb{R})$
$\mathbb{C} \oplus \mathbb{C}$
$\mathbb{R}^4$
$\alpha, -$
$\alpha, \pm$
$\beta, +$
$\mathbb{O}$
$\mathbb{O}_s$
?
?
?
?
$\mathbb{R}^8$

Thank you!