$*$-algebra structures on path algebras

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Motivation from noncommutative geometry

In noncommutative geometry, one represents a geometric space by an algebra (of functions), and in the differential subbranch one often assumes this algebra to be a $*$-algebra (or even a C^*-algebra).

Previously, Arnlind (2024) has studied the geometry of the path algebra of the Kronecker quiver. Motivated by the work of Arnlind, we would like to study the noncommutative differential geometry of quiver path algebras in greater generality.

Then the question of existence of a $*$-algebra structure on path algebras arises. Is there some structure on the quiver that is in "correspondence" with $*$-algebra structures on the path algebra?

Quivers and path algebras

A *quiver* is a directed graph $Q = (Q_0, Q_1, t, h)$ where Q_0 is the set of *vertices*, Q_1 is the set of *arrows*, and for an arrow $\alpha \in Q_1$ the vertice $t(\alpha)$ is the *tail* and $h(\alpha)$ is the *head*, i.e. $\alpha : t(\alpha) \to h(\alpha)$. Its *opposite quiver* Q^{op} switches the places of $t(\alpha)$ and $h(\alpha)$. We assume the sets Q_0, Q_1 to be finite.

A *path* p in Q is a sequence of arrows $p = \alpha_n \dots \alpha_1$ such that $t(\alpha_i) = h(\alpha_{i-1})$. We also include a path of length zero e_i for each vertex $i \in Q_0$.
The *path algebra* $\mathbb{C}Q$ of Q is the vector space

$$\mathbb{C}Q = \bigoplus_{p \text{ path in } Q} \mathbb{C}p$$

with multiplication defined by concatenation of paths where possible and zero elsewhere. The path algebra is an associative algebra and it is finite dimensional if and only if the quiver contains no cycles.

Example

Consider the quiver $Q : 1 \xrightarrow{\alpha} 2$. The path algebra of Q is generated by $\{e_1, e_2, \alpha\}$ with multiplication table

	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_1	α
α	α	0	0

The path algebra $\mathbb{C}Q$ is isomorphic to the algebra of lower triangular 2×2 matrices $L_2(\mathbb{C}) = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} \end{pmatrix}$ via

$$e_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \alpha \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$*$-algebras

A *$*$-algebra* $(A, *)$ is a associative unital $\mathbb{C}$-algebra A together with a map $* : A \to A$ satisfying

$$(a^*)^* = a, \quad (\lambda a + b)^* = \overline{\lambda} a^* + b^*, \quad (ab)^* = b^* a^*, \quad 1_A^* = 1_A$$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

Example

- The full matrix algebra $M_n(\mathbb{C})$ with the conjugate transpose is a $*$-algebra.
- The algebra of lower triangular 2×2 matrices $L_2(\mathbb{C})$ with $*$-structure defined on generators by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Quasi–symmetric quivers

When can a path algebra $\mathbb{C}Q$ be given a $*$-structure?

Note that if $\mathbb{C}Q$ is a $*$-algebra, then the $*$-structure is an (anti-linear) isomorphism of algebras $\mathbb{C}Q \to \mathbb{C}Q^{op}$. We want a corresponding structure on quivers.

A *quasi–symmetric quiver* is a quiver Q together with a map $\sigma : Q_0 \sqcup Q_1 \to Q_0 \sqcup Q_1$ satisfying

(i) $\sigma^2 = \mathrm{id}$.

(ii) $\sigma(Q_0) = Q_0$ and $\sigma(Q_1) = Q_1$.

(iii) $\sigma(h\alpha) = t\sigma(\alpha)$ and $\sigma(t\alpha) = h\sigma(\alpha)$ for all $\alpha \in Q_1$.

In other words, a quasi–symmetry is an involutive isomorphism $\sigma : Q \xrightarrow{\sim} Q^{op}$ of directed graphs.

Quasi–symmetric quivers

Example

(i) Consider the quiver $Q : 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$, and let $\sigma(1) = 2$ and $\sigma(\alpha) = \beta$.

Then (Q, σ) is a quasi–symmetric quiver:

$$\left[2 \underset{\alpha}{\overset{\beta}{\rightrightarrows}} 1\right] \simeq \left[1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2\right]$$

(ii) Given a quiver $Q = (Q_0, Q_1, t, h)$, recall the extended quiver $\widehat{Q} = (Q_0, Q_1 \sqcup Q_1^*, t, h)$, where $Q_1^* = \{\alpha^* : h(\alpha) \to t(\alpha) \mid \alpha \in Q_1\}$. Defining σ by

$$\sigma(i) = i, \quad \sigma(\alpha) = \alpha^*, \quad \text{and} \quad \sigma(\alpha^*) = \alpha, \quad i \in Q_0, \quad \alpha \in Q_1,$$

then $(\widehat{Q}, \sigma)$ is a quasi–symmetric quiver.

$*$-path algebras

Proposition

If (Q, σ) is a quasi–symmetric quiver then $\mathbb{C}Q$ can be equipped with a $*$-structure.

Proof idea.

Define $e_i^* := e_{\sigma(i)}$ and $(\alpha_n \dots \alpha_1)^* := \sigma(\alpha_1) \dots \sigma(\alpha_n)$. □

A C^**-algebra* is a triple $(A, *, ||\cdot||)$ where $(A, *)$ is a $*$-algebra, $(A, ||\cdot||)$ is a Banach algebra, and $||a^*a|| = ||a||^2$.

Proposition

Let (Q, σ) be a quasi–symmetric quiver and $\mathbb{C}Q$ its induced $*$-path algebra. If $\mathbb{C}Q$ admits a positive definite norm $||\cdot|| : \mathbb{C}Q \to \mathbb{C}$ such that $||a^*a|| = ||a||^2$ then for every arrow $\alpha : i \to j$ there exists an arrow $\alpha^{op} : j \to i$.

$*$-path algebras

Are all $*$-structure on path algebras induced from quasi–symmetries on the underlying quiver? No.

Non–Example

Consider the quiver $Q : 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$, and define $*$ by

$$\alpha^* = \beta, \quad e_1^* = e_2 - \alpha - \beta, \quad \text{and} \quad e_2^* = e_1 + \alpha + \beta.$$

Then $\mathbb{C}Q$ is a $*$-algebra, e.g.

$$(e_1^*)^* = e_2^* - \alpha^* - \beta^* = e_1 + \alpha + \beta - \beta - \alpha = e_1,$$
$$e_1^* \alpha^* = (e_2 - \alpha - \beta)\beta = \beta = \alpha^* = (\alpha e_1)^*,$$
$$e_2^* \beta^* = (e_1 + \alpha + \beta)\alpha = 0 = (\beta e_2)^*,$$

but $*(Q_0) \neq Q_0$. Note, however, that Q can still be equipped with a quasi–symmetry.

Inducing a quasi–symmetry

Theorem

Let Q be an quiver. Then $\mathbb{C}Q$ admits a $*$-algebra structure if and only if Q admits a quasi–symmetric structure.

Proof idea.

The quiver Q can be reconstructed from $\mathbb{C}Q$: a simple module S_i corresponds to a node i, and the number of arrows $i \to j$ is exactly $\dim \operatorname{Ext}^1(S_i, S_j)$. The $*$-structure induces an equivalence functor

$$H_A : \operatorname{Mod} \mathbb{C}Q \xrightarrow{\sim} \operatorname{Mod} \mathbb{C}Q^{op}$$

which preserves simple modules and dimensions of Ext–groups. □

Future investigations: $*$-structure from quasi–symmetry from $*$-structure

$$(\mathbb{C}Q, *) \longrightarrow (Q, \sigma) \longrightarrow (\mathbb{C}Q, *_\sigma)$$

Example

Recall the quiver $Q : 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$, and the $*$-structure

$$\alpha^* = \beta, \quad e_1^* = e_2 - \alpha - \beta, \quad \text{and} \quad e_2^* = e_1 + \alpha + \beta.$$

Following the construction

$$\sigma_1 : 1 \leftrightarrow 2, \quad \alpha \leftrightarrow \alpha, \quad \beta \leftrightarrow \beta, \quad \text{or} \quad \sigma_2 : 1 \leftrightarrow 2, \quad \alpha \leftrightarrow \beta.$$

Question

Is every $*$-algebra $(\mathbb{C}Q, *)$ $*$-isomorphic to a $*$-algebra on the form $(\mathbb{C}Q, \sigma)$?

Future investigations: Representations of quasi–symmetric quivers

In the representation theory of finite–dimensional (unital associative) algebras, a classic result is that the module category mod A of a finite–dimensional algebra A is equivalent to the category of representations $\text{rep}(Q, I)$ of a (bound) quiver (Q, I).

For a $*$-algebra A, a *$*$-representation* is an A-module M together with a nondegenerate hermitian $\mathbb{C}$-sesquilinear form $\langle \cdot, \cdot \rangle : M \oplus M \to \mathbb{C}$ such that

$$\langle a \cdot m, n \rangle = \langle m, a^* \cdot n \rangle,$$

for all $a \in A$ and $m, n \in M$.

Question

Given a (quasi–)symmetric quiver (Q, σ), is it possible to find a correspondence between the "representations of (Q, σ)" and the $*$-representations of the $*$-path algebra $(\mathbb{C}Q, \sigma)$? Is it then possible to extend the classical result to finite–dimensional $*$-algebras?

References

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- H. Derksen and J. Weyman. *Generalized Quivers Associated to Reductive Groups*. Colloq. Math., 94, 2002.

Tack!