

# On $(p, q)$ -palintropic algebras

Vladimir G. Tkachev

(based on a joint work with D.J. Fox, Universidad Politécnica de Madrid)

- 1 Preliminaries and medial algebras
- 2 Algebras with unconstrained Peirce spectrum
- 3 Some unexpected applications

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# Preliminaries

- $\mathbb{k}$  is a field of characteristic not equal to 2 and 3;
- $(\mathbb{A}, *)$  is a **commutative nonassociative** algebra with multiplication  $*$  over  $\mathbb{k}$
- $L(x)y := x * y$ ;
- $\text{Idm}(\mathbb{A})$  is the set of nonzero algebra idempotents;
- $\text{Idm}^\times(\mathbb{A})$  is the set of algebra idempotents  $c$  with invertible  $L(c)$ ;
- $\mathbb{A}_c(\lambda) = \ker(L(c) - \lambda\text{Id})$  is the  $\lambda$ -Peirce subspace

# Medial algebras

A commutative algebra  $\mathbb{A}$  is called **medial** if it associates on pairs:

$$M(x, y, z, w) := (xy)(zw) - (xz)(yw) = 0. \quad (1)$$

- ① It follows immediately from (1) that if  $c \in \text{Idm}(\mathbb{A})$  then  $y = w = c$  yields

$$(cx)(cz) = (xz)(cc) = c(xz) \quad \Rightarrow \quad L(c) \text{ is an algebra homomorphism.}$$

- ② in particular,

$$\mathbb{A}_c(\lambda)\mathbb{A}_c(\mu) \subset \mathbb{A}_c(\lambda\mu) \quad (2)$$

- ③ If a medial algebra  $\mathbb{A}$  is **unital** with units  $e$  then setting  $w = e$  yields

$$(xy)z - (xz)y = 0 \quad \Rightarrow \quad \mathbb{A} \text{ is associative}$$

- ④ If  $c_1, c_2$  are idempotents in a medial algebra  $\mathbb{A}$  then so is  $c_1c_2$ . In other words, the set of all idempotents is a multiplicative magma.

# Medial vs associative

## Lemma 1

Let  $c \in \text{Idm}^\times(\mathbb{A})$ . Define

$$x \bullet y = L(c)^{-1}(xy).$$

Then the new algebra  $(\mathbb{A}, \bullet)$  is associative.

Indeed,  $(\mathbb{A}, \bullet)$  is medial:

$$(x \bullet y) \bullet (z \bullet w) = L(c)^{-1}(xy) \bullet L(c)^{-1}(zw) = L(c)^{-2}((xy)(zw))$$

and unital:

$$x \bullet c = L(c)^{-1}(xc) = L(c)^{-1}L(c)(x) = x,$$

thus by the above it is associative.

**Definition.** Given an algebra  $(\mathbb{A}, \bullet)$  and an algebra homomorphism  $h$ , we define a new algebra  $(\mathbb{A}, \bullet_h)$  on the vector space  $\mathbb{A}$  by

$$x \bullet_h y = h(x) \bullet h(y) = h(x \bullet y). \quad (3)$$

$(\mathbb{A}, \bullet_h)$  is called a **(weak) inner isotope** of  $(\mathbb{A}, \bullet)$ .

Denote by  $\mathfrak{M}$  (respectively by  $\mathfrak{A}$ ) the class of calibrated special commutative medial algebras (respectively calibrated commutative associative unital algebras).

## Theorem 2 (Tkachev, 2024 [2])

*The functor  $\Phi : \mathfrak{A} \rightarrow \mathfrak{M}$  given by  $\Phi(\mathbb{A}, \diamond, e, h) = (\mathbb{A}, *, e)$ , where  $x * y = h(x \diamond y)$  and  $e \in \text{Idm}^\times(\mathbb{A}, *, e)$ , is a category isomorphism. The inverse functor  $\Psi = \Phi^{-1}$  is given by  $\Psi(\mathbb{A}, *, c) = (\mathbb{A}, \diamond, c, h)$ , where  $x \diamond y = L_*(c)^{-1}(x * y)$ ; then  $c$  is a unit in  $(\mathbb{A}, \diamond)$  and  $h = L_*(c) \in \text{Aut}(\mathbb{A}, \diamond)$ .*

### Theorem 3 (Tkachev, 2024 [2])

Let a polynomial  $P \in \mathbb{k}[z]$  split over  $\mathbb{k}$  and have  $n = \deg P$  distinct roots. Then

- ①  $\mathbb{k}[z]/P \cong \mathbb{k}^n$ ;
- ②  $\text{Aut}(\mathbb{k}[z]/P) \cong \text{Aut}(\mathbb{k}^n) \cong S_n$ , the symmetric group  $S_n$ ;
- ③ an *inner isotope*  $(\mathbb{k}^n, \sigma)$  for  $\sigma \in S_n$  is a medial algebra;
- ④  $(\mathbb{k}^n, \sigma) \cong (\mathbb{k}^n, \tau)$  if and only if  $\sigma$  and  $\tau$  conjugate in  $S_n$ .

(See example below)

## Example ( $n = 3$ and $\tau = (231) \in S_3$ )

Let  $\mathbb{k}$  contain a primitive root  $\varepsilon^7 = 1$  of unit of order 7. Then

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_2y_2, x_3y_3, x_1y_1).$$

$$\text{Idm}(\mathbb{k}^3, \tau) = \{c_k := (\varepsilon^{4k}, \varepsilon^{2k}, \varepsilon^k) : k \in \mathbb{Z}/7\mathbb{Z}\}. \quad (4)$$

The multiplication rule between idempotents:

$$c_i c_j = c_{4(i+j)} \quad (5)$$

The characteristic polynomials are given by

$$\det(\lambda I - L(c_k)) = \lambda^3 - 1. \quad (6)$$

Therefore  $(\mathbb{k}^3, \tau)$  is **isospectral** [1] with the following fusion rules:

	$\mathbb{A}_c(1)$	$\mathbb{A}_c(\varepsilon)$	$\mathbb{A}_c(\varepsilon^2)$	
$\mathbb{A}_c(1)$	$\mathbb{A}_c(1)$	$\mathbb{A}_c(\varepsilon)$	$\mathbb{A}_c(\varepsilon^2)$	
$\mathbb{A}_c(\varepsilon)$	$\mathbb{A}_c(\varepsilon)$	$\mathbb{A}_c(\varepsilon^2)$	$\mathbb{A}_c(1)$	
$\mathbb{A}_c(\varepsilon^2)$	$\mathbb{A}_c(\varepsilon^2)$	$\mathbb{A}_c(1)$	$\mathbb{A}_c(\varepsilon)$	(7)

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# The Peirce polynomial

The key idea is to decompose an algebra relative to an idempotent  $c$  into bigger 'blocks'  $\mathbb{A} = \bigoplus_{\alpha \in \sigma(c)} \mathbb{A}_c(\alpha)$  and study the **fusion** rules between the blocks

$$\mathbb{A}_c(\alpha)\mathbb{A}_c(\beta) \subset \bigoplus_{\gamma \in \rho(\alpha, \beta)} \mathbb{A}_c(\gamma),$$

where

$$\mathbb{A}_\alpha = \ker(L(c) - \alpha \text{Id})$$

$\sigma(c)$  is the Peirce spectrum of  $c$  and  $\rho : \sigma(c) \times \sigma(c) \rightarrow 2^{\sigma(c)}$ .

For example, fusion rules of a Jordan algebra or a power-associative algebra:

$\star$	$1$	$0$	$\frac{1}{2}$
$1$	$1$	$\emptyset$	$\frac{1}{2}$
$0$	$\emptyset$	$0$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$1, 0$

## Example 1

An algebra is power-associative if  $\langle\langle x \rangle\rangle$  is associative. (Albert, 1948): if  $\mathbb{A}$  is commutative and  $\text{char } \mathbb{k} \nmid 30$  then the latter is equivalent to

$$P(x) := x^4 - x^2x^2 = 0 \quad (8)$$

The linearization yields

$$2L(x)^3 + L(x)L(x^2) - 4L(x^2)L(x) + L(x^3) = 0$$

If  $c \in \text{Idm}(\mathbb{A})$  then (10) becomes

$$\underbrace{2L(c)^3 - 3L(c)^2 + L(c)}_{\text{the Peirce polynomial}} = \underbrace{(2L(c) - 1)(L(c) - 1)L(c)}_{\text{implies the Peirce spectrum}} = 0$$

**Peirce decomposition:**

$$\mathbb{A} = \mathbb{A}_c(\mathbf{1}) \oplus \mathbb{A}_c(\mathbf{0}) \oplus \mathbb{A}_c\left(\frac{1}{2}\right).$$

# $(p, q)$ -palintropic algebras

An algebra identity is called **evanescent** if its Peirce polynomial is identically zero.

## Definition

Given two integers  $2 \leq p \leq q$ , an algebra satisfying

$$(x^p)^q = (x^q)^p.$$

is called  $(p, q)$ -palintropic.

## Remark

The terminology goes back to Etherington (1945-1948): an algebra is called *palintropic* if  $(x^\alpha)^\beta = (x^\beta)^\alpha$  for *any* nonassociative powers  $\alpha$  and  $\beta$ .

## (2, 3)-palintropic algebras

We shall consider only (2, 3)-palintropic algebras, i.e. satisfying

$$(x^2x^2)x^2 - x^3x^3 = (x^2)^3 - (x^3)^2 = 0. \quad (9)$$

The linearization yields

$$L(x^3)L(x^2) + 2L(x^3)L^2(x) - 2L^2(x^2)L(x) - L(x^2x^2)L(x) = 0 \quad (10)$$

If  $c \in \text{Idm}(\mathbb{A})$  then (10) becomes

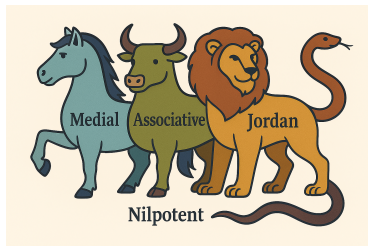
$$0 = 0 \dots \quad (11)$$

thus, the Pierce spectrum of such an algebra is **unconstrained**.

## Proposition 1

The following algebras are  $(2, 3)$ -palintropic:

- ① any algebra  $\mathbb{A}$  with  $\mathbb{A}^3 = 0$ ;
- ② any commutative power-associative algebra,
- ③ any Jordan algebra (if  $\text{char } \mathbb{k}$  is prime to 2);
- ④ any medial algebra:  $(xy)(zw) - (xz)(yw) = 0$ ;
- ⑤ any  $J$ -medial algebra:  $(xy)(x^2z) - (x^2y)(xz) = 0$ ;
- ⑥ any prepalintropic algebra:  $x^3(x^2y) - (x^2x^2)(xy) = 0$ ,



## Theorem 4

Let  $\mathbb{A}$  be  $(2, 3)$ -palintropic and let  $c \in \text{Idm}(\mathbb{A})$ . If  $\lambda, \mu \in \sigma(c) \setminus \{\frac{1}{2}\}$  then

$$\mathbb{A}_c(\lambda)\mathbb{A}_c(\mu) \subset \mathbb{A}_c(\lambda\mu). \quad (12)$$

Furthermore,

- ①  $\mathbb{A}_c(0)$  and  $\mathbb{A}_c(1)$  are subalgebras of  $\mathbb{A}$ .
- ② If  $c$  is a semi-simple idempotent,  $\frac{1}{2} \notin \sigma(c)$  then  $\mathbb{A}_c(0)$  is an ideal of  $\mathbb{A}$ .
- ③ If  $x \in \mathbb{A}_c(\lambda)$  and  $\lambda \neq \frac{1}{2}$  then

$$cx^\delta = \lambda^{\deg(x^\delta)}x^\delta$$

for any nonassociative power  $x^\delta$ .

- ④ **But:** if  $x \in \mathbb{A}_c(\lambda)$  and  $\lambda \neq \frac{1}{2}$  and  $\lambda \neq 0$  then

$$x^4 = \lambda x^2 x^2. \quad (13)$$

## Theorem 5

- ① Any *unital*  $(2,3)$ -palintropic algebra over a field of  $\text{char } \mathbb{k} \nmid 30$  is *power-associative*.
- ② In particular, the subalgebra  $\mathbb{A}_c(1)$  is power-associative.

## Theorem 6 (Almost mediality)

Suppose that  $\mathbb{A}$  is an  $(2, 3)$ -palintropic algebra and  $c \in \text{Idm}(\mathbb{A})$  is such that  $\frac{1}{2} \notin \sigma(e)$ . Then  $L(c)$  is an algebra homomorphism.

**Idea of the proof:** the 3rd linearisation yields

$$\begin{aligned} (x^2y)^2 + 4(x^2y)(x(xy)) + 4x^3(y(yx)) + 2x^3(xy^2) + 4(x(xy))^2 \\ - 8(x^2(xy))(xy) - 4x^2(xy)^2 - (x^2x^2)y^2 - 2x^2(x^2y^2) = 0. \end{aligned} \quad (14)$$

If  $y = c$  is an algebra idempotent, we have from (14)

$$(cy)^2 - 4(cy)(c(cy)) + 4(c(cy))^2 + 4c(y(yc)) - 4c(cy)^2 - cy^2 = 0$$

and therefore  $(cy - 2c(cy))^2 = c(y - 2cy)^2$  implying

$$(L(c)\phi(y))^2 = L(c)(\phi(y)^2), \quad (15)$$

where  $\phi(y) := y - 2cy$ . Note that if  $\frac{1}{2} \notin \sigma(e)$  then  $\phi$  is bijective, hence polarizing (15) we arrive at  $L(c)(xy) = (L(c)x)(L(c)y)$ .

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# $(p, q)$ -palintropic algebras and integrable polynomial maps

Motivated by integrability of discrete analogue of commuting flows (Veselov 1991).

**Definition.** A polynomial map of degree  $\geq 2$   $P(u) : \mathbb{k}^n \rightarrow \mathbb{k}^n$  is *integrable* if there exists a polynomial map  $Q(u) : \mathbb{k}^n \rightarrow \mathbb{k}^n$  of degree  $\geq 2$  such that  $P \circ Q = Q \circ P$  and the sets of iterates of  $P$  and  $Q$  are disjoint.

The well-known example is the Chebyshev polynomials of the 1st kind:

$$T_n(\cos t) = \cos nt,$$

$$T_1 = z, \quad T_2 = 2z^2 - 1, \quad T_3 = 4z^3 - 3z, \dots$$

Generic polynomial automorphisms DO NOT commute. Known examples of commuting polynomial maps of degree  $\geq 2$  are linked to elliptic curves or algebraic group structures using [simple Lie algebras and Weyl group symmetries](#) (Veselov, Chalykh, Hoffman etc) and also related to [orthogonal polynomials in several variables](#) (Koomwinder, Eier, Lidl).

# $(p, q)$ -palintropic algebras and integrable polynomial maps

Let  $\Delta$  be an equilateral triangle. The associated group of reflections has presentation  $\langle r_1, r_2, r_3 \mid r_i^2 = (r_i r_{i+1})^3 = 1 \rangle$ . If we think of  $\mathbb{R}^2$  as the set of points  $(t_1, t_2, t_3) \in \mathbb{R}^3$  with  $t_1 + t_2 + t_3 = 0$ , then this group acts on  $\mathbb{R}^2$  by

$$r_1 : t_1 \leftrightarrow t_2, \quad r_2 : t_2 \leftrightarrow t_3, \quad r_3(t_1, t_2, t_3) = (t_3 + 1, t_2, t_1 - 1), \quad (16)$$

The the generalized cosine map is given by

$$h(\mathbf{t}) = (\phi(t_1) + \phi(t_2) + \phi(t_3), \phi(t_1 + t_2) + \phi(t_2 + t_3) + \phi(t_3 + t_1)),$$

where  $\phi(t) = e^{-2\pi it}$ . Then  $h$  is invariant under (16). Then polynomials  $A_n$  associated with  $\Delta$  are defined by

$$A_k(w) = h(kw), \quad w \in \mathbb{R}^2 \cong \mathbb{C}, \quad k = 1, 2, 3, \dots$$

$$A_k \circ A_m = A_m \circ A_k.$$

$$A_1 = z, \quad A_2 = z^2 - 2\bar{z}, \quad A_3 = z^3 - 3z\bar{z} + 3, \dots \text{ or in coordinates:}$$

$$A_1 = (x, y), \quad A_2 = (-2x + x^2 - y^2, 2y + 2xy), \dots$$

# $(p, q)$ -palintropic algebras and integrable polynomial maps

Now, one can think of the  $(p, q)$ -palintropic algebra identity  $(x^p)^q - (x^q)^p = 0$  as the commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\phi_p} & \mathbb{A} \\ \phi_q \downarrow & & \downarrow \phi_q \\ \mathbb{A} & \xrightarrow{\phi_p} & \mathbb{A} \end{array}$$

where  $\phi_k(x) = x^k$  is the  $k$ th principal power. Then writing  $x^p$  in a basis we get a commutative pair. For example,  $(x^2)^3 = (x^3)^2$  generates two polynomial maps  $P_2$  and  $P_3$ , where  $P_2^{\circ k}$  is of even degree map, while  $P_3^{\circ m}$  has odd degree, so that they don't have common terms, thus both maps are integrable in the above sense.

## Example 7

A toy example in  $\mathbb{k}^3$  from a medial 3D algebra

$$p(s) = (2\alpha x, 2\beta y, 2\alpha\beta z + 2xy),$$

$$q(s) = (\alpha(2\alpha + 1)x, \beta(2\beta + 1)y, \alpha\beta(2\alpha\beta + 1)z + 2(\alpha\beta + \alpha + \beta)xy)$$

**THANK YOU FOR YOUR ATTENTION!**

# Bibliography I

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